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# BALANCED TRUNCATION MODEL REDUCTION FOR SYMMETRIC SECOND ORDER SYSTEMS—A PASSIVITY-BASED APPROACH*

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**Abstract.** We introduce a model reduction approach for linear time-invariant second order systems based on positive real balanced truncation. Our method guarantees to preserve asymptotic stability and passivity of the reduced order model as well as the positive definiteness of the mass and stiffness matrices. Moreover, we receive an a priori gap metric error bound. Finally we show that our method based on positive real balanced truncation preserves the structure of overdamped second order systems.

**Key words.** model reduction, matrix polynomials, balanced truncation, indefinite linear algebra

**AMS subject classifications.** 46C20, 93C, 93C15

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## 1. Introduction.

In this article, we consider linear second order systems with co-located inputs and outputs of the form

$$M\ddot{\xi}(t) + D\dot{\xi}(t) + K\xi(t) = Bu(t), \quad y(t) = B^\top \dot{\xi}(t) \tag{1.1}$$

with symmetric $M, D, K \in \mathbb{R}^{n\times n}$, where the *mass matrix* $M$ and the *stiffness matrix* $K$ are positive definite, while the *damping matrix* $D$ is positive semidefinite and $B \in \mathbb{R}^{n\times m}$. In applications, the state space dimension $n$ typically becomes unfeasibly large for simulation, optimization, or control. This causes a demand for a good approximation of the system. Therefore, an upper bound in a certain quality measure is desirable for a model order reduction method. Such a quality measure can be some norm of the error system, such as the $H^\infty$ norm for stable systems, or the so-called *gap metric* [13, 14] between the original and reduced order system. The latter expresses the distance between the input-output trajectories of two systems. On the other hand, our model has three key attributes that a reduced model should inherit, one of these being the second order structure together with the symmetries, meaning the reduced system should be of the form

$$\widetilde{M}\ddot{\widetilde{\xi}}(t) + \widetilde{D}\dot{\widetilde{\xi}}(t) + \widetilde{K}\widetilde{\xi}(t) = \widetilde{B}u(t), \quad \widetilde{y}(t) = \widetilde{B}^\top \dot{\widetilde{\xi}}(t), \tag{1.2}$$

with symmetric $\widetilde{M}$, $\widetilde{D}$, $\widetilde{K} \in \mathbb{R}^{r\times r}$, and $\widetilde{B} \in \mathbb{R}^{r\times m}$, where $r \ll n$. Other physically meaningful properties are asymptotic stability and passivity, which should also be preserved. Passivity describes the property that the energy in the system is solely induced by input and output and may be dissipated or conserved in the system.

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Some progress has been made in [7, 33] and preservation of physical properties in the reduced order model is possible for co-located inputs and outputs. Further second order reduction methods have been developed in [3, 7, 9, 12, 21, 28, 36] (see also [33] for an overview). Besides these, there exist interpolatory methods, which succeed in either preserving the second order structure [37] or delivering a posteriori $H^\infty$ error bounds [29]. However, all the approaches mentioned lack a combination of the two.

These approaches all have in common that the ansatz for the reduced order model is $\xi(t) = W_r\widehat{\xi}(t)$ for some "tall matrix" $W_r \in \mathbb{R}^{n\times r}$ and an accordant multiplication of the state equation in (1.2) from the left with some "flat" matrix $V_r \in \mathbb{R}^{r\times n}$. Our approach is somewhat different from these. Namely, we first perform a reduction of the first order representation, and accordingly carry out a transformation yielding a second order model. This corresponds to a reduction ansatz $\xi(t) = W_{r,1}\widetilde{\xi}(t) + W_{r,2}\dot{\widetilde{\xi}}(t)$ together with a linear combination of the state equation and its derivative such that again a second order system is obtained. The basis for our considerations is the technique of *positive real balanced truncation*, which is a passivity-preserving method for first order systems; see [10]. Moreover, an a priori error bound in the gap metric is provided [19]. We present a modification of positive real balanced truncation, which preserves the second order structure, symmetry of the system matrices, passivity as well as positive definiteness of the mass, and stiffness matrices. Moreover, if the original system is asymptotically stable, then the reduced system inherits this property. Namely, we show that, under an extra condition on the so-called zero sign characteristics, the reduced order model obtained by positive real balanced truncation of a first order realization of (1.1) can be transformed to a second order realization of the form (1.2). To this end, we use the theory of *standard triples* by Gohberg, Lancaster, and Rodman [16] to derive a second order realization (1.2) in which the mass and stiffness matrices are positive definite, and the damping matrix has at least $r-m$ positive eigenvalues provided that $r \geq m$. Furthermore, if the original system is *overdamped* (see (7.1)), our method will further produce a reduced order model with positive definite damping matrix.

**Our method in a nutshell.** The model reduction technique presented in this article consists—in theory—of three steps:

*Step* 1: For $H, G \in \mathbb{R}^{n\times n}$ with $M = HH^\top$, $K = GG^\top$, we set $x(t) := \left[\begin{smallmatrix} G^\top H^{-\top}\widehat{\xi}(t) \\ \dot{\widehat{\xi}}(t)\end{smallmatrix}\right]$ with $\widehat{\xi}(t) := H^\top\xi(t)$ and rewrite the second order system (1.1) as

(1.3)
$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & G^\top H^{-\top} \\ -H^{-1}G & -H^{-1}DH^{-\top}\end{bmatrix}}_{=:\mathcal{A}} x(t) + \underbrace{\begin{bmatrix} 0 \\ H^{-1}B\end{bmatrix}}_{=:\mathcal{B}} u(t), \quad y(t) = \underbrace{\begin{bmatrix} 0 & B^\top H^{-\top}\end{bmatrix}}_{=:\mathcal{C}} x(t).$$

The most important feature of this system is that it has an internal symmetry structure $\mathcal{A}\mathscr{S}_n = \mathscr{S}_n\mathcal{A}^\top$ and $\mathcal{C} = \mathcal{B}^\top = \mathcal{B}^\top\mathscr{S}_n$, where $\mathscr{S}_n := \operatorname{diag}(-I_n, I_n)$. In particular, its *transfer function* $\mathbf{G}(s) = \mathcal{C}(sI_{2n} - \mathcal{A})^{-1}\mathcal{B}$ is symmetric, i.e., it fulfills $\mathbf{G}(s)^\top = \mathbf{G}(s)$. We will make heavy use of this symmetry structure.

*Step* 2: We apply positive real balanced truncation [19] to the first order system (1.3). The internal symmetry structure of (1.3) yields that positive real balanced truncation can be done by determining only one (instead of two) solutions of the Kálmán–Yakubovich–Popov inequality. We show that the positive real characteristic values (see Definition 2.2(b)), can—in a certain sense—be allocated to the symmetry structure of the system (1.3). This is the basis for our finding that the resulting first order model is—without making any further computational effort—of the form

$$
\begin{aligned}
\dot{\widetilde{x}}(t) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \mathcal{A}_{16} \\ 0 & 0 & 0 & 0 & \mathcal{A}_{25} & \mathcal{A}_{26} \\ 0 & 0 & \mathcal{A}_{33} & \mathcal{A}_{34} & 0 & \mathcal{A}_{36} \\ 0 & 0 & -\mathcal{A}_{34}^\top & \mathcal{A}_{44} & 0 & \mathcal{A}_{46} \\ 0 & -\mathcal{A}_{25}^\top & 0 & 0 & 0 & 0 \\ -\mathcal{A}_{16}^\top & -\mathcal{A}_{26}^\top & -\mathcal{A}_{36}^\top & \mathcal{A}_{46}^\top & 0 & \mathcal{A}_{66} \end{bmatrix} \widetilde{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{B}_6 \end{bmatrix} u(t), \\
\widetilde{y}(t) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \mathcal{B}_6^\top \end{bmatrix} \widetilde{x}(t),
\end{aligned} \tag{1.4}
$$

where the block sizes from left to right and from top to bottom are $m, \ell, p, p, \ell, m$, with $r = p + m + \ell$. Note that if $\mathcal{A}_{33}$ is zero, then it would—by merging some of the blocks—be of the form

$$
\begin{aligned}
\dot{\widetilde{x}}(t) &= \begin{bmatrix} 0 & \widetilde{G}^\top \\ -\widetilde{G} & -\widetilde{D} \end{bmatrix} \widetilde{x}(t) + \begin{bmatrix} 0 \\ \widetilde{\mathcal{B}} \end{bmatrix} u(t), \\
\widetilde{y}(t) &= \begin{bmatrix} 0 & \widetilde{\mathcal{B}}^\top \end{bmatrix} \widetilde{x}(t),
\end{aligned} \tag{1.5}
$$

which would result in a reduced second order model (1.2) with $\widetilde{M} = I_r$, $\widetilde{K} = \widetilde{G}\widetilde{G}^\top$, and $\widetilde{D} = \widetilde{D}^\top$. This is regrettably not the case in general, hence we apply the following step.

*Step* 3: We apply a state space transformation to (1.4) such that the matrix $\mathcal{A}_{33}$ vanishes. More precisely, we first intend to find some $\widehat{T} \in \mathrm{Gl}_{2p}(\mathbb{R})$ that preserves the symmetry structure, i.e., it fulfills $\widehat{T}^\top \mathscr{S}_p \widehat{T} = \mathscr{S}_p$, and

$$
\widehat{T}^{-1} \begin{bmatrix} \mathcal{A}_{33} & \mathcal{A}_{34} \\ -\mathcal{A}_{34}^\top & \mathcal{A}_{44} \end{bmatrix} \widehat{T} = \begin{bmatrix} 0 & \widehat{\mathcal{A}}_{34} \\ -\widehat{\mathcal{A}}_{34}^\top & \widehat{\mathcal{A}}_{44} \end{bmatrix}. \tag{1.6}
$$

Then a state space transformation with $T = \mathrm{diag}(I_{m+\ell}, \widehat{T}, I_{\ell+m})$ leads to a system which is indeed of the form (1.5) and can then be rewritten as a second order system. To derive such a transformation, we have to use techniques from indefinite linear algebra. In particular, the aforementioned symmetry structure of our system provides that each real (invariant) zero of the system can be assigned a signature; see Definition 4.6. If the reduced system is minimal, the zeros with positive signature are given by some $\mu_1^+ \leq \cdots \leq \mu_k^+ < \mu_{k+1}^+ = \cdots = \mu_{k+m}^+ = 0$ and those with negative signature by $\mu_1^- \leq \cdots \leq \mu_k^- < 0$. Our main results on the existence and construction of such transformations are the following (see Theorem 6.7). Suppose the reduced system is minimal; then the following are equivalent:
(a) The system (1.4) can be written in second order form (1.2).
(b) There exists a $\widehat{T} \in \mathrm{Gl}_{2p}(\mathbb{R})$ with $\widehat{T}^\top \mathscr{S}_p \widehat{T} = \mathscr{S}_p$ that fulfills (1.6).
(c) For $i = 1, \ldots, k$ it holds that $\mu_i^- < \mu_i^+$.
Here, the implication "(a)$\Rightarrow$(c)" is based on [26, Thm. 16]. If the reduced system is not minimal or (c) is not fulfilled, we add equations to the system in a fashion that the newly formed system is also stable, passive, and such that the transfer function is preserved. Here, in the worst case the number of states, i.e., the size of the mass matrix, doubles.

**Outline of the article.** The article is structured as follows. In section 2 we introduce the passivity preserving balanced truncation and some background material from systems theory. In section 3 we present our positive real balanced truncation ansatz for second order systems and prove some of its main features. The derivation of the balanced form of second order systems, namely (1.4), is done in section 5. Before we focus on the reconstruction of the second order structure and the proof of the necessary condition on the zeros in section 6, section 4 introduces some necessary concepts from indefinite linear algebra. Section 7 considers the case that the original system is overdamped. Last but not least, in section 8 we summarize our method in a numerical procedure and also discuss the numerical treatment.

**Notation.** The set of natural numbers including zero is denoted by $\mathbb{N}$. The symbols $\mathbb{R}[s]$ and $\mathbb{R}(s)$ respectively stand for the ring of real polynomials and the field of real rational functions. We denote by $\mathbb{C}^+$ and $\overline{\mathbb{C}^+}$ the open and closed complex half plane. Further, $\mathbf{G}(s) \in \mathbb{R}(s)^{m\times m}$ is called *proper* if $\lim_{s\to\infty} \mathbf{G}(s) < \infty$ and *strictly proper* if the latter limit is zero. The transpose and conjugate transpose of $T \in \mathbb{C}^{m\times n}$ are denoted by $T^\top$ and $T^*$, respectively. For symmetric matrices $X, Y \in \mathbb{R}^{n\times n}$, we write $X > Y$ if $X - Y$ is positive definite and $X \geq Y$ if $X - Y$ is positive semidefinite. The symbol $\mathrm{Gl}_n(\mathbb{R})$ stands for the set of invertible $n \times n$ matrices with entries in $\mathbb{R}$. We call $S = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_i = \pm 1$ for $i = 1, \ldots, n$, a *signature matrix* and make frequent use of the signature matrix $\mathscr{S}_n := \mathrm{diag}(-I_n, I_n) \in \mathbb{R}^{2n\times 2n}$. For $A \in \mathbb{C}^{m\times n}$ and $x \in \mathbb{C}^n$, $\|A\|$ and $\|x\|$ respectively stand for the maximum singular value of $A$ and the Euclidean norm of $x$.

## 2. Positive real balanced truncation of first order systems.

We revisit positive real balanced truncation for linear time-invariant first order systems

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad y(t) = \mathcal{C}x(t) + \mathcal{D}u(t) \tag{2.1}$$

with $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B}, \mathcal{C}^\top \in \mathbb{R}^{n\times m}$, and $\mathcal{D} \in \mathbb{R}^{m\times m}$. The dynamical system (2.1) is said to be *minimal* if it is both controllable and observable. The *transfer function* is given by $\mathbf{G}(s) = \mathcal{C}(sI_n - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D} \in \mathbb{R}(s)^{m\times m}$. We also speak of (2.1) as a realization of $\mathbf{G}(s)$ and use the notation $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ to refer to this system, or if $\mathcal{D} = 0$ we write $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$. We call $\mu \in \mathbb{C}$ an *(invariant) zero of* $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ if

$$\mathrm{rank}_{\mathbb{C}} \begin{bmatrix} -\mu I_n + \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} < \mathrm{rank}_{\mathbb{R}(s)} \begin{bmatrix} -sI_n + \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}.$$

In other words, the set of zeros of $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ equals to the set of eigenvalues of the matrix pencil $\left[\begin{smallmatrix} -sI_n + \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$. If the latter pencil is square and invertible as a matrix over $\mathbb{R}(s)$ (which is, by taking the Schur complement, equivalent to the transfer function being invertible), then $\mu \in \mathbb{C}$ is a zero of $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$, if and only if there exists some $v \in \mathbb{C}^{n+m} \setminus \{0\}$ with $\left[\begin{smallmatrix} -\mu I_n + \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] v = 0$.

In the following we deal with systems having *positive real* transfer functions. That is,

(a) $\mathbf{G}(s)$ has no poles in $\mathbb{C}^+$,
(b) $\mathbf{G}(\lambda) + \mathbf{G}(\lambda)^* \geq 0$ for all $\lambda \in \mathbb{C}^+$.

If $\mathbf{G}(s)$ is positive real and invertible, also $\mathbf{G}^{-1}(s)$ is positive real. As a consequence, realizations $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ of positive real functions with the property that all eigenvalues of $\mathcal{A}$ have nonpositive real part do not have any zeros in $\mathbb{C}^+$. In particular, minimal realizations of positive real transfer functions do not have any zeros in $\mathbb{C}^+$. The famous positive real lemma draws a link between positive realness of the transfer function and the solvability of a certain linear matrix inequality.

THEOREM 2.1 (positive real lemma [2, Chap. V]). *Let* $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ *be a minimal system of the form* (2.1) *with transfer function* $\mathbf{G}(s) \in \mathbb{R}(s)^{m\times m}$. *Then* $\mathbf{G}(s)$ *is* *positive real if and only if there exists some* $P > 0$*, such that the* Kálmán–Yakubovich–Popov (KYP) inequality

$$\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) := \begin{bmatrix} \mathcal{A}^\top P + P\mathcal{A} & P\mathcal{B} - \mathcal{C}^\top \\ \mathcal{B}^\top P - \mathcal{C} & -\mathcal{D} - \mathcal{D}^\top \end{bmatrix} \leq 0 \tag{2.2}$$

*is fulfilled. Moreover, there exists a* minimal *solution* $P_{\min} > 0$ *and a* maximal *solution* $P_{\max} > 0$ *of* $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) \leq 0$*, i.e., for all other solutions* $P$ *of* (2.2) *it holds that* $P_{\max} \geq P \geq P_{\min}$*.*

The KYP inequality admits the so-called dissipation inequality. That is, for all locally square integrable solutions $(x, u, y)$ of $[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]$ and $t > 0$ it holds that

$$\tfrac{1}{2}x(t)^\top Px(t) \leq \tfrac{1}{2}x(0)^\top Px(0) + \int_0^t y(\tau)^\top u(\tau)\,\mathrm{d}\tau\,. \tag{2.3}$$

Such systems are also called *passive*. Since $\mathbf{G}(s)^\top$ is the transfer function of $[\mathcal{A}^\top, \mathcal{C}^\top, \mathcal{B}^\top, \mathcal{D}^\top]$, this system is passive as well. Therefore, the dual KYP inequality $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{C}^\top,\mathcal{B}^\top,\mathcal{D}^\top]}(Q) \leq 0$ has again a minimal solution. Moreover, $P > 0$ solves $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) \leq 0$ if and only if $P^{-1}$ is a solution of $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{C}^\top,\mathcal{B}^\top,\mathcal{D}^\top]}(Q) \leq 0$. As a consequence, if $P_{\min}$ is the minimal such solution in the sense of Theorem 2.1, then $P_{\min}^{-1}$ is the maximal solution of the dual KYP inequality.

Definition 2.2 (positive real balanced, internally passive). *With the notation of Theorem* 2.1*, a system* $[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]$ *is called*

(a) positive real balanced *if for the minimal solutions* $P_{\min}$*,* $Q_{\min}$ *of the KYP inequalities* $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) \leq 0$ *and* $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{C}^\top,\mathcal{B}^\top,\mathcal{D}^\top]}(Q) \leq 0$ *we have*

$$P_{\min} = Q_{\min} = \operatorname{diag}(\sigma_1 I_{n_1}, \ldots, \sigma_h I_{n_h}),$$

*where* $\sigma_1, \ldots, \sigma_h$ *are distinct values in* $(0, 1]$*; the latter are called the* positive real characteristic values *of* $[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]$*;*

(b) internally passive *if* $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(I_n) \leq 0$.

In contrast to the conventional definition of positive real balanced, we do not assume that the positive real characteristic values of $[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]$ are ordered.

Note that internal passivity implies that in the dissipation inequality (2.3), the quadratic form with $P$ is the square of the norm of the state. It has been shown in Theorem 7 of [34] that any positive real balanced realization is internally passive.

If there exist minimal solutions $P_{\min} \geq 0$, $Q_{\min} \geq 0$ of the KYP inequalities $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) \leq 0$ and $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{C}^\top,\mathcal{B}^\top,\mathcal{D}^\top]}(Q) \leq 0$, then a certain state space transformation leads to a positive real balanced realization. We refer to this as *positive real balancing*. Our main emphasis is on *positive real balanced truncation*, that is, balancing the system and accordingly removing the blocks corresponding to some positive real characteristic values. This leads to a reduced system which is passive and, if the transfer function of the original system is asymptotically stable, is again asymptotically stable; see [30, Thm. 3.2]. The proof of the latter theorem is valid also for positive real balanced truncation as discussed in [20, sect. III]. Both steps can be done at once while also removing the uncontrollable and unobservable parts [38]. Namely, by using factorizations $P_{\min} = L^\top L$, $Q_{\min} = R^\top R$ and the singular value decomposition

$$LR^\top = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \tag{2.4}$$

we are able to define the reduction matrices $W^\top = \Sigma_1^{-1/2} Z_1 L$ and $V = R^\top U_1 \Sigma_1^{-1/2}$. A reduced model received from positive real balanced truncation is given by the system $[W^\top \mathcal{A} V, W^\top \mathcal{B}, \mathcal{C}V, \mathcal{D}]$. If we do not truncate any singular values, then we obtain a minimal system, as the following lemma shows.

LEMMA 2.3. *Let a system $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ with positive real transfer function $\mathbf{G}(s)$ be given. Assume that $P_{\min}$, $Q_{\min}$ are minimal solutions of the KYP inequalities $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) \leq 0$ and $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{C}^\top,\mathcal{B}^\top,\mathcal{D}^\top]}(Q) \leq 0$, and let $L$ and $R$ be matrices with full row rank and $P_{\min} = L^\top L$, $Q_{\min} = R^\top R$. Further, let $LR^\top = U\Sigma Z$ be a singular value decomposition and let $W^\top = \Sigma^{-1/2} Z L$, $V = R^\top U \Sigma^{-1/2}$. Then $[W^\top \mathcal{A} V, W^\top \mathcal{B}, \mathcal{C}V, \mathcal{D}]$ is a minimal and positive real balanced realization of $\mathbf{G}(s)$. Moreover, for any further minimal positive real balanced realization $[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}, \widehat{\mathcal{C}}, \mathcal{D}]$ of $\mathbf{G}(s)$ with positive real characteristic values in the same order, there exist orthogonal matrices $Q_i \in \mathbb{R}^{n_i \times n_i}$ for $i = 1, \dots, h$ such that for $T := \operatorname{diag}(Q_1, Q_2, \dots, Q_h)$, $\widehat{\mathcal{A}} = T^\top \mathcal{A} T$, $\widehat{\mathcal{B}} = T^\top \mathcal{B}$, and $\widehat{\mathcal{C}} = \mathcal{C}T$.*

*Proof.* By using an appropriate state space transformation, we can assume that $P_{\min} =: \operatorname{diag}(P_1, 0)$ for some positive definite matrix $P_1$, and partition $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ accordingly, i.e., $\mathcal{A} = \left[\begin{smallmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{smallmatrix}\right]$, and $\mathcal{C} = \left[\begin{smallmatrix} \mathcal{C}_1 & \mathcal{C}_2 \end{smallmatrix}\right]$. Since $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]} \leq 0$, the block form yields $\mathcal{A}_{12} = 0$ and $\mathcal{C}_2 = 0$, i.e., we obtain a Kálmán observability decomposition in which the kernel of $P$ corresponds to the unobservable states. On the other hand, if $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ is in Kálmán observability decomposition, and $P_{\min} = \left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{smallmatrix}\right]$ according to the block structure of the Kálmán controllability decomposition, then a simple calculation shows that $P = \left[\begin{smallmatrix} P_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]}(P) \leq 0$. The minimality of $P_{\min}$ therefore leads to $P_{12} = 0$ and $P_{22} = 0$. As a consequence, the kernel of $P$ indeed corresponds to the space of unobservable states. Likewise, the image of $Q_{\min}$ corresponds to the space of controllable states, whence, by using the results from [38], $[W^\top \mathcal{A} Z, W^\top \mathcal{B}, \mathcal{C} Z, \mathcal{D}]$ is a minimal positive real balanced realization of $\mathbf{G}(s)$.

The second statement is Lemma 6 from [34]. ∎

A consequence of this lemma is that the reduced transfer function does not depend on the specific minimal positive real balanced realization of the original transfer function.

Now we present details on the error bound of positive real balanced truncation. To this end, consider the space $\mathcal{R}H^\infty(\mathbb{C}^{m\times m})$ of proper elements of $\mathbb{R}(s)^{m\times m}$ with entries having no poles in $\overline{\mathbb{C}^+}$. Moreover, let $\mathcal{R}H^2(\mathbb{C}^m)$ the space of strictly proper elements of $\mathbb{R}(s)^m$ with entries having no poles in $\overline{\mathbb{C}^+}$. A *(right) coprime factorization* of $\mathbf{G}(s) \in \mathbb{R}(s)^{p\times m}$ is $\left[\begin{smallmatrix} \mathbf{M}(s) \\ \mathbf{N}(s) \end{smallmatrix}\right]$ consisting of $\mathbf{N}(s) \in \mathcal{R}H^\infty(\mathbb{C}^{p\times m})$, $\mathbf{M}(s) \in \mathcal{R}H^\infty(\mathbb{C}^{m\times m})$ such that $\mathbf{G}(s) = \mathbf{N}(s)\mathbf{M}(s)^{-1}$ and there exist $\mathbf{X}(s) \in \mathcal{R}H^\infty(\mathbb{C}^{m\times m})$ and $\mathbf{Y}(s) \in \mathcal{R}H^\infty(\mathbb{C}^{m\times p})$ that satisfy the *Bézout identity* $\mathbf{X}(s)\mathbf{M}(s) + \mathbf{Y}(s)\mathbf{N}(s) = I_m$. A coprime factorization $\left[\begin{smallmatrix} \mathbf{M}(s) \\ \mathbf{N}(s) \end{smallmatrix}\right]$ is called *normalized* if additionally, $\mathbf{M}^\top(-s)\mathbf{M}(s) + \mathbf{N}^\top(-s)\mathbf{N}(s) = I_m$. Such factorizations can be computed using techniques as in [27, 41]. Considering normalized coprime factorizations, a distance measure for general transfer functions can be introduced. To this end, consider the space $H^2(\mathbb{C}^m)$ of all analytic functions mapping from $\mathbb{C}^+$ to $\mathbb{C}^m$ with the property that the supremum over all integrals of the square of the norm on right parallels of the imaginary axis is bounded. The square root of the aforementioned supremum scaled by $\frac{1}{\sqrt{2\pi}}$ indeed defines a norm on $H^2(\mathbb{C}^m)$; see [43, sect. 4.2].

DEFINITION 2.4. *Let* $\mathbf{G}_1(s), \mathbf{G}_2(s) \in \mathbb{R}(s)^{p\times m}$ *be given with respective normalized coprime factorizations* $\left[\begin{smallmatrix}\mathbf{M}_1(s)\\ \mathbf{N}_1(s)\end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix}\mathbf{M}_2(s)\\ \mathbf{N}_2(s)\end{smallmatrix}\right]$. *Let* $\Pi_1, \Pi_2 : H^2(\mathbb{C}^{m+p}) \to H^2(\mathbb{C}^{m+p})$ *be orthogonal projectors with*

$$\operatorname{im}\Pi_1 = \begin{bmatrix}\mathbf{M}_1(s)\\ \mathbf{N}_1(s)\end{bmatrix} \cdot H^2(\mathbb{C}^m), \qquad \operatorname{im}\Pi_2 = \begin{bmatrix}\mathbf{M}_2(s)\\ \mathbf{N}_2(s)\end{bmatrix} \cdot H^2(\mathbb{C}^m).$$

*Then the* gap between $\mathbf{G}_1(s)$ and $\mathbf{G}_2(s)$ *is defined via*

$$\delta_g(\mathbf{G}_1(s), \mathbf{G}_2(s)) := \|\Pi_1 - \Pi_2\|_{L(H^2(\mathbb{C}^{m+p}))},$$

*where* $\|\cdot\|_{L(H^2(\mathbb{C}^{m+p}))}$ *denotes the operator norm on* $H^2(\mathbb{C}^{m+p})$.

It is shown in [40] that $\delta_g(\cdot,\cdot)$ fulfills the axioms of a metric. The gap metric between two (possibly unstable) systems is simply the gap metric between their transfer functions. The above avowedly abstract definition of the gap metric has a rather tangible interpretation in terms of the distance between the trajectories of two systems $[\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_i]$, $i = 1, 2$, which have the same input dimension $m$ and the same output dimension $p$: Consider the *space of stable input-output trajectories with trivial initial value*, that is,

$$\begin{aligned}\mathfrak{B}_i = \{(u_i, y_i) \in \mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^{m+p}) \mid \exists x_i \in \mathcal{H}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^{n_i})&\\ \text{such that } \dot{x}_i = A_i x_i + B_i u_i,\ y_i = C_i x_i + D_i u_i\},&\end{aligned}$$

where $\mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^k)$ stands for Lebesgue space of square integrable $\mathbb{R}^k$-valued equivalence classes of functions defined on $\mathbb{R}_{\geq 0}$, and $\mathcal{H}^1(\mathbb{R}_{\geq 0}, \mathbb{R}^k)$ is the space of elements of $\mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^k)$ which possess a weak derivative in $\mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^k)$. It is proven in [4] that for any $(u_1, y_1) \in \mathfrak{B}_1$, there exists some $(u_2, y_2) \in \mathfrak{B}_2$ such that

$$\|u_1 - u_2\|_2^2 + \|y_1 - y_2\|_2^2 \leq \delta_g(\mathbf{G}_1(s), \mathbf{G}_2(s))^2 \cdot (\|u_1\|_2^2 + \|y_1\|_2^2),$$

where $\mathbf{G}_i(s) \in \mathbb{R}(s)^{p\times m}$ is the transfer function of $[\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_i]$, and $\|\cdot\|_2$ stands for the norm in the Lebesgue space $\mathcal{L}^2(\mathbb{R}_{\geq 0}, \mathbb{R}^k)$.

Positive real balanced truncation has the following gap metric error bound.

THEOREM 2.5 (see [19, Cor. 2.2]). *Let* $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ *be a realization of the positive real function* $\mathbf{G}(s) \in \mathbb{R}(s)^{m\times m}$. *Denote the positive real characteristic values by* $(\sigma_i)_{i=1}^h$ *and, for* $r < h$, *let* $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}, \mathcal{D}]$ *be obtained by a positive real balanced truncation of* $[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}]$ *by removing the blocks corresponding to* $\sigma_{r+1}, \dots, \sigma_h$. *Then the transfer function* $\widetilde{\mathbf{G}}(s)$ *of* $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}, \mathcal{D}]$ *fulfills*

$$\delta_g(\mathbf{G}(s), \widetilde{\mathbf{G}}(s)) \leq 2 \sum_{i=r+1}^{h} \sigma_i.$$

Note that [19] considers positive real balanced truncation in which the states corresponding to the smallest characteristic values are removed. A careful inspection of the proof of the error bound (and those of the results used therein) yields that the above error bound still holds when states corresponding to arbitrary positive real characteristic values are removed. In our method for second order systems we will make use of this fact.

**3. Positive real balanced truncation for second order systems.** We introduce positive real balanced truncation for systems having a symmetric second order structure. Now we consider linear time-invariant first order systems $[\mathcal{A},\mathcal{B},\mathcal{C}]$ with $\mathcal{A}\in\mathbb{R}^{2n\times 2n}$ and $\mathcal{B}\in\mathbb{R}^{2n\times m}$ structured as in (1.3), that is,

$$\mathcal{A}=\begin{bmatrix}0 & G^\top\\ -G & -D\end{bmatrix}\in\mathrm{Gl}_{2n}(\mathbb{R}),\quad \mathcal{B}=\begin{bmatrix}0\\ B\end{bmatrix}=\mathcal{C}^\top \tag{3.1}$$

for some $D\in\mathbb{R}^{n\times n}$ with $D=D^\top\geq 0$ and $B\in\mathbb{R}^{n\times m}$. The transfer function is given by

$$\mathbf{G}(s)=\mathcal{C}(sI_{2n}-\mathcal{A})^{-1}\mathcal{B}=sB^\top(s^2I_n+sD+K)^{-1}B\in\mathbb{R}(s)^{m\times m},$$

where $K=GG^\top$. We summarize these structural assumption on the system class as follows.

Definition 3.1. *We say that a system $[\mathcal{A},\mathcal{B},\mathcal{C}]$ is an element of $\Xi_{n,m}$ and write $[\mathcal{A},\mathcal{B},\mathcal{C}]\in\Xi_{n,m}$ if the following is satisfied:*

(a) *The matrices $\mathcal{A}\in\mathbb{R}^{2n\times 2n}$ and $\mathcal{B}\in\mathbb{R}^{2n\times m}$ are structured as in* (3.1) *for some $G\in\mathrm{Gl}_n(\mathbb{R})$, $D\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, and $\mathcal{C}=\mathcal{B}^\top$.*
(b) *The matrix $B$ has full column rank and $D=D^\top$.*
(c) *$[\mathcal{A},\mathcal{B},\mathcal{C}]$ is stabilizable and passive.*

We first notice the following.

*Remark* 3.2 (second order systems, passivity, and zeros).

(a) The assumption that $\operatorname{rank}B=\operatorname{rank}\mathcal{B}=m$ is no restriction, since otherwise, there exists an orthogonal matrix $T\in\mathbb{R}^{m\times m}$ such that $\mathcal{B}T=\begin{bmatrix}\mathcal{B}_1 & 0\end{bmatrix}$, where, for $\mathcal{B}_1\in\mathbb{R}^{2n\times k}$, $\operatorname{rank}\mathcal{B}_1=k$. Hence $T^\top\mathbf{G}(s)T=\left[\begin{smallmatrix}\mathbf{G}_1(s) & 0\\ 0 & 0\end{smallmatrix}\right]$ for some invertible $\mathbf{G}_1(s)\in\mathbb{R}(s)^{k\times k}$. In this case one can approximate $\mathbf{G}_1(s)$ instead and afterward add the zero rows and columns to the reduced transfer function.
(b) If $D\geq 0$, it can be seen that $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{B}^\top,0]}(I_{2n})\leq 0$ and hence $\mathbf{G}(s)$ is positive real by the positive real lemma (see Theorem 2.1). Then [32, Thm. 15] guarantees the existence of respective minimal solutions $P_{\min},Q_{\min}\geq 0$ of $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{B}^\top,0]}(P)\leq 0$ and $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{B},\mathcal{B}^\top,0]}(Q)\leq 0$ if $(\mathcal{A},\mathcal{B})$ and $(\mathcal{A}^\top,\mathcal{C}^\top)$ are stabilizable. By using the symmetry structure of the system (3.1), i.e., $\mathcal{A}^\top=\mathscr{S}_n\mathcal{A}\mathscr{S}_n$ and $\mathcal{B}^\top=\mathcal{C}\mathscr{S}_n$ for $\mathscr{S}_n=\operatorname{diag}(-I_n,I_n)$, the latter two properties are, however, equivalent due to $(\mathcal{A}^\top,\mathcal{C}^\top)=(\mathscr{S}_n\mathcal{A}\mathscr{S}_n,\mathscr{S}_n\mathcal{B})$. Since, further, $\mathcal{A}$ does not have any eigenvalues in $\mathbb{C}^+$, the absence of uncontrollable purely imaginary eigenvalues is sufficient for the existence of minimal solutions $P_{\min},Q_{\min}\geq 0$.
(c) The assumption that $\operatorname{rank}B=m$ furthermore implies that the transfer function of $[\mathcal{A},\mathcal{B},\mathcal{C}]$ with matrices in (3.1) is invertible. Further note that positive realness of $\mathbf{G}(s)$ together with $\mathcal{A}$ having no eigenvalues in $\mathbb{C}^+$ implies that the system $[\mathcal{A},\mathcal{B},\mathcal{C}]$ has no zeros in $\mathbb{C}^+$.

For systems $[\mathcal{A},\mathcal{B},\mathcal{C}]\in\Xi_{n,m}$ we begin with our reduction method pointing out the following observation. The symmetry structure further implies that $P\geq 0$ solves $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},0]}(P)\leq 0$ if and only if $Q:=\mathscr{S}_nP\mathscr{S}_n$ solves $\mathscr{W}_{[\mathcal{A}^\top,\mathcal{C}^\top,\mathcal{B}^\top,0]}(Q)\leq 0$. In particular, $Q_{\min}=\mathscr{S}_nP_{\min}\mathscr{S}_n$ and thus, for $L^\top L=P_{\min}$, we obtain that $\mathscr{S}_nL^\top L\mathscr{S}_n=Q_{\min}$. Altogether, instead of the singular value decomposition (2.4) we can compute the eigendecomposition

$$L\mathscr{S}_nL^\top=\underbrace{\begin{bmatrix}U^- & U^+\end{bmatrix}}_{=:U}\mathscr{S}_n\underbrace{\begin{bmatrix}\Sigma^- & 0\\ 0 & \Sigma^+\end{bmatrix}}_{=:\Sigma}\begin{bmatrix}U^- & U^+\end{bmatrix}^\top, \tag{3.2}$$

where $U \in \mathbb{R}^{n\times n}$ is orthogonal and

$$
\begin{aligned}
\Sigma^- &= \operatorname{diag}\left(\sigma_1^- I_{n_1^-}, \dots, \sigma_{h^-}^- I_{n_{h^-}^-}\right), &\qquad 0 \le \sigma_{h^-}^- < \dots < \sigma_1^- \le 1,\\
\Sigma^+ &= \operatorname{diag}\left(\sigma_{h^+}^+ I_{n_{h^+}^+}, \dots, \sigma_1^+ I_{n_1^+}\right), &\qquad 0 \le \sigma_{h^+}^+ < \dots < \sigma_1^+ \le 1.
\end{aligned} \tag{3.3}
$$

Next we choose some positive real characteristic values of positive and negative type which correspond to truncated states. To this end, let $r^+, r^- \in \mathbb{N}$ be such that for some $q^+, q^- \in \mathbb{N}$, $r^\pm = \sum_{j=1}^{q^\pm} n_j^\pm$. Additionally, these numbers have to be chosen such that the states corresponding to zero characteristic values are truncated, and the set of characteristic values corresponding to the truncated states does not intersect with the set of characteristic values corresponding to the preserved states. This means that

$$
\begin{gathered}
1 \le q^\pm \le h^\pm, \qquad \sigma_{q^\pm}^\pm > 0,\\
\sigma_{q^-+j^-}^- \ne \sigma_{i^+}^+ \quad\text{and}\quad \sigma_{q^++j^+}^+ \ne \sigma_{i^-}^- \ \forall\, i^\pm = 1,\dots,q^\pm,\ j^\pm = 1,\dots,h^\pm - q^\pm.
\end{gathered} \tag{3.4}
$$

Note that the above condition is only of a pathological nature and does not impose any serious restriction from a numerical point of view, since generically, it holds that the characteristic values in $(0,1)$ are simple.

The general purpose is that the reduced system can be transformed into a second order system. To this end we require that the reduced system has a symmetry structure with respect to a matrix $\mathscr{S}_r$. Hence it is essential that we find $r^\pm$ which additionally fulfill $r^- = r^+ =: r$. We partition (3.2) as

$$
L\mathscr{S}_n L^\top = \begin{bmatrix} U_1^- & U_2 & U_1^+ \end{bmatrix} \begin{bmatrix} -\Sigma_1^- & 0 & 0\\ 0 & \mathscr{S}_{n-r}\Sigma_2 & 0\\ 0 & 0 & \Sigma_1^+ \end{bmatrix} \begin{bmatrix} (U_1^-)^\top \\ (U_2)^\top \\ (U_1^+)^\top \end{bmatrix}, \tag{3.5}
$$

where $U_1^\pm \in \mathbb{R}^{2n\times r}$, $U_2 \in \mathbb{R}^{2n\times 2(n-r)}$, $\Sigma_2 \in \mathbb{R}^{2(n-r)\times 2(n-r)}$, and $\Sigma_1^\pm \in \mathbb{R}^{r\times r}$. We set $\Sigma_1 := \operatorname{diag}(\Sigma_1^-, \Sigma_1^+)$ and $U_1 := \begin{bmatrix} U_1^- & U_1^+ \end{bmatrix}$. Using the reduction matrices

$$
W^\top := \Sigma_1^{-\frac{1}{2}} \mathscr{S}_r U_1^\top L \quad\text{and}\quad V := \mathscr{S}_n L^\top U_1 \Sigma_1^{-\frac{1}{2}}, \tag{3.6}
$$

we construct the reduced first order model

$$
[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}] := [W^\top \mathcal{A} V, W^\top \mathcal{B}, \mathcal{C} V]. \tag{3.7}
$$

Next, we state some important properties of the above reduced order model.

THEOREM 3.3. *Let a system $[\mathcal{A}, \mathcal{B}, \mathcal{C}] \in \Xi_{n,m}$ be given with $D = D^\top \ge 0$. Consider the reduced system $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ as constructed as in* (3.5)–(3.7) *for some $r^+, r^- \in \mathbb{N}$ which fulfills* (3.2)–(3.4)*, in particular $r^+ = r^- =: r$. Then the following statements are satisfied:*

(a) *We have $\mathscr{W}_{[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}, 0]}(\Sigma_1) \le 0$ and $\mathscr{W}_{[\widetilde{\mathcal{A}}^\top, \widetilde{\mathcal{C}}^\top, \widetilde{\mathcal{B}}^\top, 0]}(\Sigma_1) \le 0$. In particular, $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ is passive.*

(b) *We have $\mathscr{S}_r \widetilde{\mathcal{A}} \mathscr{S}_r = \widetilde{\mathcal{A}}^\top$ and $\widetilde{\mathcal{B}} = \mathscr{S}_r \widetilde{\mathcal{B}} = \widetilde{\mathcal{C}}^\top$.*

(c) *The gap metric between the transfer functions $\mathbf{G}(s)$ and $\widetilde{\mathbf{G}}(s)$ of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ and $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ can be estimated by*

$$
\delta_g(\mathbf{G}(s), \widetilde{\mathbf{G}}(s)) \le 2 \sum_{i=q^-+1}^{h^-} \sigma_i^- + 2 \sum_{i=q^++1}^{h^+} \sigma_i^+.
$$

(d) *If* $\sigma^-_{q^-+1} = 0 = \sigma^+_{q^++1}$*, i.e.,* $\Sigma_2$ *from* (3.5) *is zero, then* $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ *is a minimal positive real balanced realization of* $\mathbf{G}(s)$*.*

*Proof.* The first two statements can be inferred from the arguments in the proof of [34, Thm. 8], whereas the third part follows from Theorem 2.5 and the last statement from Lemma 2.3. ∎

The exact block structure of the reduced system, as introduced in (1.4), will be part of section 5. First we need some results from the study of indefinite linear algebra.

**4. Preliminaries from indefinite linear algebra.** Next, we introduce some notions and results from indefinite linear algebra.

DEFINITION 4.1. *Two pairs* $(S_j, A_j) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$*,* $j = 1, 2$*, consisting of a symmetric matrix* $S_j \in \mathrm{Gl}_n(\mathbb{R})$ *and an* $S_j$-self-adjoint *matrix* $A_j$*, i.e.,* $A_j^\top S_j = S_j A_j$*, are called* congruent-similar *if there exists a* $T \in \mathrm{Gl}_n(\mathbb{R})$ *such that* $T^{-1}A_1T = A_2$ *and* $T^\top S_1 T = S_2$*.*

In [17], a canonical form under congruence-similarity is given. For the sake of simplicity we will focus on diagonalizable matrices.

THEOREM 4.2 (see [17, sect. I.5, Thm. 5.3]). *Let* $(S, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$*, where* $S \in \mathrm{Gl}_n(\mathbb{R})$ *is symmetric, and* $A$ *is* $S$*-self-adjoint and diagonalizable over* $\mathbb{C}$*. Then there exists some* $T \in \mathrm{Gl}_n(\mathbb{R})$ *such that for some* $k, c \in \mathbb{N}$ *with* $2c + k = n$ *and* $\varepsilon_1, \ldots, \varepsilon_k \in \{\pm 1\}$*,* $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$*,* $\sigma_1, \ldots, \sigma_c \in \mathbb{R}$*,* $\tau_1, \ldots, \tau_c > 0$*, and*

$$\mathscr{J}_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \mathscr{J}_c := \mathrm{diag}(\underbrace{\mathscr{J}_1, \ldots, \mathscr{J}_1}_{c\ times}), \ \mathscr{P}_{\sigma_i,\tau_i} := \begin{bmatrix} \sigma_i & \tau_i \\ -\tau_i & \sigma_i \end{bmatrix},$$

*we obtain*

$$\begin{aligned} T^\top S T &= \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_k, \mathscr{J}_c), \\ T^{-1} A T &= \mathrm{diag}(\lambda_1, \ldots, \lambda_k, \mathscr{P}_{\sigma_1,\tau_1}, \ldots, \mathscr{P}_{\sigma_c,\tau_c}). \end{aligned} \tag{4.1}$$

It has been further shown in [17, sect. I.5, Thm. 5.3] that the above is a canonical form for the pair $(S, A)$ under congruence-similarity if the tuples $(\varepsilon_1, \lambda_1), \ldots, (\varepsilon_k, \lambda_k)$ and $(\sigma_1, \tau_1), \ldots, (\sigma_c, \tau_c)$ are ordered increasingly with respect to the lexicographical order. It can be seen that the eigenvalues of $A$ in Theorem 4.2 are given by $\lambda_1, \ldots, \lambda_k$ and $\sigma_1 \pm \mathrm{i}\tau_1, \ldots, \sigma_c \pm \mathrm{i}\tau_c$. Based on this normal form we derive a special form for $S$-self-adjoint and diagonalizable matrices whose eigenvalues are in the closed complex left half plane.

COROLLARY 4.3. *Let* $(S, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ *be as in Theorem* 4.2*. If all eigenvalues of* $A$ *have negative real part, then there exist* $T \in \mathrm{Gl}_n(\mathbb{R})$ *and* $c, k, k_1, k_2 \in \mathbb{N}$ *with* $k_1 + k_2 = k$ *and* $2c + k = n$ *such that*

$$T^{-1} A T = \begin{bmatrix} 0 & 0 & \mathcal{V} \\ 0 & \Lambda & 0 \\ -\mathcal{V} & 0 & \mathcal{E} \end{bmatrix}, \quad T^\top S T = \mathrm{diag}(-I_{c+k_1}, I_{k_2+c}), \tag{4.2}$$

*where* $\mathcal{E}, \mathcal{V} \in \mathbb{R}^{c\times c}$ *and* $\Lambda \in \mathbb{R}^{k\times k}$ *are negative definite and diagonal.*

*Proof.* Without loss of generality (w.l.o.g.) we can assume that $(S, A)$ is in the canonical form of Theorem 4.2. The assumption on the spectrum of $A$ implies that $\sigma_i < 0$ for $i = 1, \ldots, c$. Since further, $\tau_i > 0$ for all $i = 1, \ldots, c$, the matrix

$$\Theta_i := \frac{1}{\sqrt{2\tau_i}} \begin{bmatrix} \sqrt{-\sigma_i+\sqrt{\sigma_i^2+\tau_i^2}} & -\sqrt{-\sigma_i+\sqrt{\sigma_i^2+\tau_i^2}} \\ \sqrt{\tau_i}\left(-2\sqrt{-\sigma_i+\sqrt{\sigma_i^2+\tau_i^2}}\right)^{-1} & \sqrt{\tau_i}\left(-2\sqrt{-\sigma_i+\sqrt{\sigma_i^2+\tau_i^2}}\right)^{-1} \end{bmatrix} \tag{4.3}$$

is real and straightforward computations show that

$$\Theta_i^\top \mathscr{J}_1 \Theta_i = \mathscr{S}_1 \quad \text{and} \quad \Theta_i^{-1} \mathscr{P}_{\sigma_i,\tau_i} \Theta_i = \mathscr{S}_1 \Theta_i^\top \mathscr{S}_1 \mathscr{P}_{\sigma_i,\tau_i} \Theta_i = \begin{bmatrix} 0 & \nu_i \\ -\nu_i & \eta_i \end{bmatrix},$$

where $\eta_i = \frac{-3\sigma_i^2+4\sigma_i\sqrt{\sigma_i^2+\tau_i^2}}{-2\sigma_i+2\sqrt{\sigma_i^2+\tau_i^2}} < 0$. Setting $T := \operatorname{diag}(I_k, \Theta_1, \dots, \Theta_c)$ and suitably interchanging the rows and columns of the tuple $(T^\top ST, T^{-1}AT)$ leads to the form (4.2). ☐

The form (4.2) plays a key role later in section 6. By starting from a realization where some blocks containing the system zeros have this form, under certain conditions we are able to bring the reduced system into the form (1.3) and thereby retrieve the second order structure.

DEFINITION 4.4. *Let $(S, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ be as in Theorem 4.2. Regarding the canonical form in Theorem 4.2, we call the tuple $((\varepsilon_1, \lambda_1), \dots, (\varepsilon_k, \lambda_k))$ the* sign characteristics *of $(S, A)$. Further we call an eigenvalue $\lambda$ of $A$ an eigenvalue of $(S, A)$, and we say that a real eigenvalue $\lambda$ of $(S, A)$ is of* positive (negative) *type if $(1, \lambda)$ $((-1, \lambda))$ is contained in the sign characteristics of $(S, A)$.*

We will often indicate that an eigenvalue is of positive (negative) type by equipping it with a superscript "+" ("−"). Note that $\lambda$ can be of both negative and positive type. The next result follows rather directly from the definition of the sign characteristics.

PROPOSITION 4.5. *Let $(S, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ be as in Theorem 4.2. An eigenvalue $\lambda \in \mathbb{R}$ of $(S, A)$ is of positive (negative) type if and only if there exists an eigenvector $v \in \mathbb{R}^n \setminus \{0\}$ of $A$ corresponding to the eigenvalue $\lambda$ of $A$ such that $v^\top Sv > 0$ $(v^\top Sv < 0)$.*

*Proof.* This is obviously true if $(S, A)$ is in the canonical form (4.1). The general statement then follows by a transformation of $(S, A)$ into this canonical form. ☐

We are now able to define the pole and zero sign characteristics of a system $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$. Recall that the zeros of a system $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ are the eigenvalues of the pencil $\left[\begin{smallmatrix} -sI_n+\mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{smallmatrix}\right]$. If the transfer function is square and invertible, we call a zero $\mu$ of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ *semisimple* if it is a semisimple eigenvalue of the latter pencil. That is, the order of the zero $\mu$ of $\det\left[\begin{smallmatrix} -sI_n+\mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{smallmatrix}\right]$ equals the dimension of the kernel of the complex matrix $\left[\begin{smallmatrix} -\mu I_n+\mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{smallmatrix}\right]$.

DEFINITION 4.6. *Let a system $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ with invertible transfer function $\mathbf{G}(s) \in \mathbb{R}^{m\times m}$ be given such that, for a signature matrix $S \in \mathrm{Gl}_n(\mathbb{R})$, it holds that $S\mathcal{A}S = \mathcal{A}^\top$ and $\mathcal{B} = S\mathcal{B} = C^\top$. We say that such a system is* internally symmetric *(w.r.t. $S$). We call the sign characteristics of $(S, \mathcal{A})$ the* pole sign characteristics *of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$. Further suppose that the zeros of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ are semisimple and denote the real zeros of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ by $\mu_1, \dots, \mu_k$. For $i = 1, \dots, k$, let $\left[\begin{smallmatrix} v_i \\ w_i \end{smallmatrix}\right] \in \ker\left[\begin{smallmatrix} -\mu_i I_n+\mathcal{A} & B \\ \mathcal{C} & 0 \end{smallmatrix}\right]$. Then we call*

$$\left(\left(-\operatorname{sign}(v_1^\top Sv_1), \mu_1\right), \dots, \left(-\operatorname{sign}(v_k^\top Sv_k), \mu_k\right)\right)$$

*the* zero sign characteristics *of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$. We say that an eigenvalue $\lambda \in \mathbb{R}$ of $\mathcal{A}$ is a pole of positive (negative) type of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ if $(1, \lambda)$ $((-1, \lambda))$ is contained in the pole* *sign characteristics of* $[\mathcal{A},\mathcal{B},\mathcal{C}]$. *Similarly, we say that a zero* $\mu\in\mathbb{R}$ *of* $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *is a zero of positive (negative) type of* $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *if* $(1,\mu)$ $((-1,\mu))$ *is contained in the zero sign characteristics of* $[\mathcal{A},\mathcal{B},\mathcal{C}]$.

Straightforward calculations show that the system $[\mathcal{A},\mathcal{B},\mathcal{C}]$ with the properties as in Definition 4.6 has a symmetric transfer function. On the other hand, note that the notions of pole and zero sign characteristics are defined for symmetric and invertible transfer functions in [17, sect. II.3.2] by means of pole and zero sign characteristics of a minimal realization. It is further shown that these are well-defined in the sense that they do not depend on the minimal realization of a given symmetric and invertible transfer function. The basis for this is that, by the results in [17, sect II.3.2], for any realization $[\mathcal{A},\mathcal{B},\mathcal{C}]$ of a symmetric and invertible transfer function $\mathbf{G}(s)\in\mathbb{R}(s)^{m\times m}$, there exists a unique nonsingular Hermitian matrix $S$ with

$$S\mathcal{A}=\mathcal{A}^\top S,\qquad S\mathcal{B}=\mathcal{C}^\top,\qquad \mathcal{C}=\mathcal{B}^\top S. \tag{4.4}$$

Further, for two minimal realizations $[\mathcal{A}_i,\mathcal{B}_i,\mathcal{C}_i]$, $i=1,2$, of $\mathbf{G}(s)$, with Hermitian matrices $S_1$ and $S_2$ from (4.4), the unique state space transformation $T\in\mathrm{Gl}_n(\mathbb{R})$ between the two realizations fulfills

$$T^{-1}=S_2^{-1}T^\top S_1. \tag{4.5}$$

**5. Positive real balanced realizations of second order systems.** The aim in this part is to prove the block structure of the reduced system from section 3, introduced in (1.4). For this purpose, we apply three successive transformations to this system. First, we derive a different first order representation of our second order system. Then we develop an input-output normal form from which one can read off the different types of system zeros, namely the real and complex ones and those on the imaginary axis. We use this form to arrive at a positive real balanced realization of the original system and deduce that the reduced model has a balanced realization of the same block structure. Later in section 6, in order to find a second order realization of the reduced system, we actually proceed conversely.

As a first step toward a positive real balanced system we consider a system of the form (3.1) with a special structure which displays the zeros on the imaginary axis.

Lemma 5.1. *Let a system* $[\mathcal{A},\mathcal{B},\mathcal{C}]\in\Xi_{n,m}$ *be given. Then the transfer function* $\mathbf{G}(s)$ *of* $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *is invertible. Moreover, there exists a state space transformation* $T=\mathrm{diag}(T_1,T_2)\in\mathrm{Gl}_{2n}(\mathbb{R})$ *with orthogonal* $T_1,T_2\in\mathrm{Gl}_n(\mathbb{R})$ *such that* $[\mathcal{A}_\mathrm{s},\mathcal{B}_\mathrm{s},\mathcal{C}_\mathrm{s}]=[T^{-1}\mathcal{A}T,T^{-1}\mathcal{B},\mathcal{C}T]$ *has the form*

$$\mathcal{A}_\mathrm{s}=\begin{bmatrix}0&0&0&0&0&G_{31}^\top\\0&0&0&0&G_{22}^\top&G_{32}^\top\\0&0&0&G_{13}^\top&0&G_{33}^\top\\0&0&-G_{13}&-D_{11}&0&-D_{13}\\0&-G_{22}&0&0&0&0\\-G_{31}&-G_{32}&-G_{33}&-D_{13}^\top&0&-D_{33}\end{bmatrix},\quad \mathcal{B}_\mathrm{s}=\begin{bmatrix}0\\0\\0\\0\\0\\B_3\end{bmatrix}=\mathcal{C}_\mathrm{s}^\top, \tag{5.1}$$

*where, for some* $\ell\in\mathbb{N}$, *the blocks in the above form are of sizes* $m$, $\ell$, $n-m-\ell$, $n-m-\ell$, $\ell$, *and* $m$. *Further,* $G_{31},B_3\in\mathbb{R}^{m\times m}$, $G_{22}\in\mathbb{R}^{\ell\times\ell}$, *and* $G_{13}\in\mathbb{R}^{(n-m-\ell)\times(n-m-\ell)}$ *are invertible. The matrix* $\left[\begin{smallmatrix}0&G_{13}^\top\\-G_{13}&-D_{11}\end{smallmatrix}\right]$ *does not have eigenvalues on the imaginary axis. If, further,* $D\geq 0$, *then those eigenvalues all have negative real part.*

*Moreover, the set of zeros of* $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *is given by the union of* $\{0\}$ *and the spectra of* $\left[\begin{smallmatrix}0&G_{13}^\top\\-G_{13}&-D_{11}\end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix}0&G_{22}^\top\\-G_{22}&0\end{smallmatrix}\right]$. *If further,* $[\mathcal{A},\mathcal{B},\mathcal{C}]$ *is minimal, then all zero sign* *characteristics of $[\mathcal{A},\mathcal{B},\mathcal{C}]$ at zero are of positive type, whereas the sign characteristics of the remaining real zeros coincide with the sign characteristics of*

$$\left(\begin{bmatrix} I_{n-m-\ell} & 0 \\ 0 & -I_{n-m-\ell}\end{bmatrix}, \begin{bmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11}\end{bmatrix}\right).$$

*Proof. Step* 1. We prove the existence of a block diagonal state space transformation $T \in \mathrm{Gl}_{2n}(\mathbb{R})$ such that $[\mathcal{A}_{\mathrm{s}},\mathcal{B}_{\mathrm{s}},\mathcal{C}_{\mathrm{s}}] = [T^{-1}\mathcal{A}T, T^{-1}\mathcal{B}, \mathcal{C}T]$ has the form (5.1) such that $B_3$, $G_{13}$, $G_{22}$, and $G_{31}$ are invertible, and $D_{11}v \neq 0$ for each eigenvector $v \in \mathbb{R}^{n-m-\ell}\setminus\{0\}$ of $G_{13}$.

Since $B$ has full column rank, we have a QR-decomposition $B = T_{21}\left[\begin{smallmatrix} 0 \\ B_3\end{smallmatrix}\right]$ with invertible $B_3$. Further, by QR-decomposition of $G^\top T_{21}$ and permutation of rows, we see that there exists some orthogonal $T_{11}\in\mathbb{R}^{n\times n}$ with

$$G^\top T_{21} = T_{11}^\top \begin{bmatrix} 0_{m\times(n-m)} & \widetilde{G}_{21}^\top \\ \widetilde{G}_{12}^\top & \widetilde{G}_{22}^\top \end{bmatrix}.$$

Now applying the state space transformation $\widehat{T}_1 = \mathrm{diag}(T_{11}, T_{21})$ to $[\mathcal{A},\mathcal{B},\mathcal{C}]$ we obtain a system of the form

$$\widehat{T}_1^\top \mathcal{A}\widehat{T}_1 = \begin{bmatrix} 0 & 0 & 0 & \widetilde{G}_{21}^\top \\ 0 & 0 & \widetilde{G}_{12}^\top & \widetilde{G}_{22}^\top \\ 0 & -\widetilde{G}_{12} & -\widetilde{D}_{11} & -\widetilde{D}_{12} \\ -\widetilde{G}_{21} & -\widetilde{G}_{22} & -\widetilde{D}_{12}^\top & -\widetilde{D}_{22}\end{bmatrix}, \quad \widehat{T}_1^\top\mathcal{B} = \left(\mathcal{C}\widehat{T}_1\right)^\top = \begin{bmatrix} 0 \\ 0 \\ 0 \\ B_3\end{bmatrix}.$$

Our next step is to apply a further state space transformation which separates the purely imaginary eigenvalues of $\left[\begin{smallmatrix} 0 & \widetilde{G}_{12}^\top \\ -\widetilde{G}_{12} & -\widetilde{D}_{11}\end{smallmatrix}\right]$. To this end, we perform a singular value decomposition $\widetilde{G}_{12} = T_{12}\widehat{G}_{12}T_{22}^\top$ with orthogonal matrices $T_{12}, T_{22}\in\mathbb{R}^{(n-m)\times(n-m)}$ and a diagonal matrix $\widehat{G}_{12}\in\mathbb{R}^{(n-m)\times(n-m)}$. Further, for each eigenspace of $\widehat{G}_{12}$, we perform an orthogonal decomposition into the intersection of this eigenspace with $\ker T_{22}^\top\widetilde{D}_{11}T_{22}$ and its orthogonal complement in this eigenspace. An accordant orthogonal transformation along with a permutation matrix leads to the existence of some orthogonal matrices $T_{23}, T_{13}\in\mathbb{R}^{(n-m)\times(n-m)}$ that lead to the following transformed matrices. First, for some $\ell\in\mathbb{N}$, $G_{22}\in\mathbb{R}^{\ell\times\ell}$, $G_{13}\in\mathbb{R}^{(n-m-\ell)\times(n-m-\ell)}$ we have

$$T_{23}^\top\widetilde{G}_{12}T_{13} = \begin{bmatrix} 0 & G_{13} \\ G_{22} & 0\end{bmatrix}.$$

Moreover, for some $D_{11}\in\mathbb{R}^{(n-m-\ell)\times(n-m-\ell)}$, $D_{13}\in\mathbb{R}^{(n-m-\ell)\times m}$, $D_{33}\in\mathbb{R}^{m\times m}$ it holds that

$$\begin{bmatrix} T_{23}^\top T_{22}^\top & 0 \\ 0 & I_m\end{bmatrix} T_{21}^\top D T_{21}\begin{bmatrix} T_{22}T_{23} & 0 \\ 0 & I_m\end{bmatrix} = \begin{bmatrix} D_{11} & 0 & D_{13} \\ 0 & 0 & 0 \\ D_{13}^\top & 0 & D_{33}\end{bmatrix}.$$

Last, for each eigenvector $v\in\mathbb{R}^{n-m-\ell}\setminus\{0\}$ of the diagonal matrix $G_{13}$ we have $D_{11}v\neq 0$ and for some $G_{31}\in\mathbb{R}^{m\times m}$, $G_{32}\in\mathbb{R}^{m\times(n-m-\ell)}$, $G_{33}\in\mathbb{R}^{m\times(n-m-\ell)}$ it holds that

$$\begin{bmatrix} T_{23}^\top T_{22}^\top & 0 \\ 0 & I_m\end{bmatrix} T_{21}^\top G T_{11}\begin{bmatrix} I_m & 0 \\ 0 & T_{12}T_{13}\end{bmatrix} = \begin{bmatrix} 0 & 0 & G_{13} \\ 0 & G_{22} & 0 \\ G_{31} & G_{32} & G_{33}\end{bmatrix}.$$

In particular, $G_{13}$, $G_{22}$, and $G_{31}$ are invertible since $G$ is invertible. Altogether, for the orthogonal matrices $T_1 = T_{11}\left[\begin{smallmatrix} I_m & 0 \\ 0 & T_{12}T_{13} \end{smallmatrix}\right]$, $T_2 = T_{21}\left[\begin{smallmatrix} T_{22}T_{23} & 0 \\ 0 & I_m \end{smallmatrix}\right]$, a state space transformation with $T = \operatorname{diag}(T_1, T_2) \in \mathrm{Gl}_{2n}(\mathbb{R})$ results in a system $[\mathcal{A}_\mathrm{s}, \mathcal{B}_\mathrm{s}, \mathcal{C}_\mathrm{s}] = [T^{-1}\mathcal{A}T, T^{-1}\mathcal{B}, \mathcal{C}T]$ which is of the form (5.1).

*Step* 2. We prove that for the construction in Step 1, $\left[\begin{smallmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{smallmatrix}\right]$ does not have eigenvalues on the imaginary axis and, provided that $D \geq 0$, those eigenvalues all have negative real part.

The fact that all eigenvalues of $\left[\begin{smallmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{smallmatrix}\right]$ have nonpositive real part if $D \geq 0$ follows from the fact that the sum of this matrix and its Hermitian is negative semidefinite. To show that in any case it does not have any eigenvalues on the imaginary axis, assume that $\omega \in \mathbb{R}$, $v_1, v_2 \in \mathbb{C}^{n-m-\ell}$ are given such that

$$\begin{bmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathrm{i}\omega \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{5.2}$$

A multiplication of (5.2) from the right with $\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)^*$ and taking the real part yields $v_2^* D_{11} v_2 = 0$, whence, by $D_{11} \geq 0$, $D_{11}v_2 = 0$. Hence

$$\begin{bmatrix} 0 & G_{13}^\top \\ -G_{13} & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathrm{i}\omega \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \ \wedge \ D_{11}v_2 = 0. \tag{5.3}$$

On the other hand, the first relation in (5.3) yields $G_{13}G_{13}^\top v_2 = -\omega^2 v_2$, whence, by the fact that $G_{13}$ is diagonal, $v_2$ is zero or an eigenvector of $G_{13}$. Then we obtain $v_2 = 0$ by the results of Step 1, and the invertibility of $G_{13}$ further gives rise to $v_1 = 0$.

*Step* 3. We show that the transfer function $\mathbf{G}(s)$ is invertible, and the set of zeros of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ is given by the union of $\{0\}$ and the spectra of $\left[\begin{smallmatrix} 0 & G_{13}^\top \\ -G_{31} & -D_{11} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & G_{22}^\top \\ -G_{22} & 0 \end{smallmatrix}\right]$.

This follows by the fact that

$$\det \begin{bmatrix} -sI_{2n} + \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{bmatrix} = c \cdot s^m \cdot \det \begin{bmatrix} sI_{n-m-\ell} & -G_{13}^\top \\ G_{13} & sI_{n-m-\ell} + D_{11} \end{bmatrix} \cdot \det \begin{bmatrix} sI_\ell & -G_{22}^\top \\ G_{22} & sI_\ell \end{bmatrix}$$

for some $c \in \mathbb{R} \setminus \{0\}$.

*Step* 4. We prove the statement about the sign characteristics.

This follows since $\left[\begin{smallmatrix} v_3 \\ v_4 \end{smallmatrix}\right] \in \mathbb{C}^{2(n-m-\ell)}$ is an eigenvector of $\left[\begin{smallmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{smallmatrix}\right]$ corresponding to the eigenvalue $\lambda \in \mathbb{C}$ if and only if there exists some $v_7 \in \mathbb{C}^m$ such that

$$\begin{bmatrix} 0 \\ 0 \\ v_3 \\ v_4 \\ 0 \\ 0 \\ v_7 \end{bmatrix} \in \ker \begin{bmatrix} -\lambda I_{2n} + \mathcal{A}_\mathrm{s} & \mathcal{B}_\mathrm{s} \\ \mathcal{C}_\mathrm{s} & 0 \end{bmatrix}.$$

The statement for the sign characteristics of the zeros at zero is completely analogous. $\square$

Starting from the second order system structured as in the above lemma, we can determine a normal form which displays the different types of zeros. In particular, we find that solutions of the KYP inequalities from the positive real lemma are block diagonal matrices structured according to the zero blocks. This helps us to derive a positive real balanced system of the form (1.4). Moreover, in section 6, we take this normal form as a basis to bring the reduced system back to second order form and to check a necessary condition whether this is actually possible. Before we present the normal form we need a small lemma.

LEMMA 5.2. *Let $Y \in \mathbb{R}^{n\times n}$ be skew-symmetric and $Z \in \mathbb{R}^{n\times n}$ be symmetric and positive semidefinite. Then $ZY + Y^\top Z \leq 0$ if and only if $ZY + Y^\top Z = 0$.*

*Proof.* Since by using $Z = Z^\top$ and $Y = -Y^\top$, an evaluation of the diagonal entries of $ZY+Y^\top Z$ yields that these vanish, $ZY+Y^\top Z \leq 0$ implies that $ZY+Y^\top Z = 0$. ∎

THEOREM 5.3. *Let a system $[\mathcal{A},\mathcal{B},\mathcal{C}] \in \Xi_{n,m}$ be given with $D \geq 0$. Suppose that the zeros of the system are semisimple. Then there exists a state space transformation $T \in \mathrm{Gl}_{2n}(\mathbb{R})$ such that the system $[\mathcal{A}_\mathrm{n},\mathcal{B}_\mathrm{n},\mathcal{C}_\mathrm{n}] := [T^{-1}\mathcal{A}T, T^{-1}\mathcal{B}, \mathcal{C}T]$ has the block form*

$$\mathcal{A}_\mathrm{n} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A}_{27} & \mathcal{A}_{28} \\ 0 & 0 & 0 & 0 & 0 & \mathcal{A}_{36} & 0 & \mathcal{A}_{38} \\ 0 & 0 & 0 & \mathcal{A}_{44} & 0 & 0 & 0 & \mathcal{A}_{48} \\ 0 & 0 & 0 & 0 & \mathcal{A}_{55} & 0 & 0 & \mathcal{A}_{58} \\ 0 & 0 & -\mathcal{A}_{36}^\top & 0 & 0 & \mathcal{A}_{66} & 0 & \mathcal{A}_{68} \\ 0 & -\mathcal{A}_{27}^\top & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathcal{A}_{18}^\top & -\mathcal{A}_{28}^\top & -\mathcal{A}_{38}^\top & -\mathcal{A}_{48}^\top & \mathcal{A}_{58}^\top & \mathcal{A}_{68}^\top & 0 & \mathcal{A}_{88} \end{bmatrix}, \quad \mathcal{C}_\mathrm{n}^\top = \mathcal{B}_\mathrm{n} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{B}_8 \end{bmatrix}, \tag{5.4}$$

*where the following hold:*

(a) *$\mathcal{A}_{18}, \mathcal{B}_8 \in \mathrm{Gl}_m(\mathbb{R})$, $\mathcal{A}_{27} \in \mathrm{Gl}_\ell(\mathbb{R})$, $\mathcal{A}_{36} \in \mathrm{Gl}_c(\mathbb{R})$, and $\mathcal{A}_{66} < 0$;*

(b) *$\mathcal{A}_{44} = \mathrm{diag}(\mu_1^+, \dots, \mu_k^+)$ and $\mathcal{A}_{55} = \mathrm{diag}(\mu_1^-, \dots, \mu_k^-)$, with $\mu_1^\pm \leq \dots \leq \mu_k^\pm < 0$. Those are the negative (real and nonzero) zeros of $[\mathcal{A},\mathcal{B},\mathcal{C}]$. If $[\mathcal{A},\mathcal{B},\mathcal{C}]$ is minimal, then, for $i = 1,\dots,k$, $\mu_i^+$ is of positive type, while $\mu_i^-$ is of negative type.*

(c) *$n = k + c + \ell + m$ and $2\ell + m$ is the number of zeros of $[\mathcal{A},\mathcal{B},\mathcal{C}]$ on $\mathrm{i}\mathbb{R}$ counted with multiplicities, while $2c$ is the number of nonreal zeros with negative real part.*

(d) *All solutions $P \geq 0$, $Q \geq 0$ of the KYP inequalities $\mathscr{W}_{[\mathcal{A}_\mathrm{n},\mathcal{B}_\mathrm{n},\mathcal{C}_\mathrm{n},0]}(P) \leq 0$ and $\mathscr{W}_{[\mathcal{A}_\mathrm{n}^\top,\mathcal{C}_\mathrm{n}^\top,\mathcal{B}_\mathrm{n}^\top,0]}(Q) \leq 0$ have the block form $P = \mathrm{diag}(I_{m+\ell}, P_2, I_{m+\ell})$ and $Q = \mathrm{diag}(I_{m+\ell}, Q_2, I_{m+\ell})$ for some $P_2, Q_2 \in \mathbb{R}^{2(c+k)\times 2(c+k)}$.*

*Proof. Step* 1. We show that there exists a state space transformation $\widetilde{T} \in \mathrm{Gl}_{2n}(\mathbb{R})$ such that $\widetilde{T}^\top \mathscr{S}_n \widetilde{T} = \mathscr{S}_n$ and $[\mathcal{A}_\mathrm{n},\mathcal{B}_\mathrm{n},\mathcal{C}_\mathrm{n}] := [\widetilde{T}^{-1}\mathcal{A}\widetilde{T}, \widetilde{T}^{-1}\mathcal{B}, \mathcal{C}\widetilde{T}]$ has the form (5.4). Moreover, we show that this realization fulfills (a)–(c) in the above theorem.

W.l.o.g. we assume that the system is already in the form $[\mathcal{A},\mathcal{B},\mathcal{C}]{=}[\mathcal{A}_\mathrm{s},\mathcal{B}_\mathrm{s},\mathcal{C}_\mathrm{s}]$ as in Lemma 5.1. In particular, all eigenvalues of $\widehat{\mathcal{A}}_\mathrm{s} {:=} \left[\begin{smallmatrix} 0 & G_{13}^\top \\ -G_{13} & -D_{11} \end{smallmatrix}\right] {\in} \mathbb{R}^{2(n-m-\ell)\times 2(n-m-\ell)}$ have negative real part. Since the nonreal eigenvalues occur in pairs and $\widehat{\mathcal{A}}_\mathrm{s} \in \mathbb{R}^{2(n-m-\ell)\times 2(n-m-\ell)}$, there exists some $c \in \mathbb{N}$ such that $2c$ is the number of nonreal eigenvalues (counted with multiplicities). Hence, the number of real eigenvalues of $\widehat{\mathcal{A}}_\mathrm{s}$ (again counted with multiplicities) is $2k$ for $k = n - c - \ell - m$. Now, according to Corollary 4.3, $k$ real eigenvalues $\mu_1^+ \leq \dots \leq \mu_k^+ < 0$ of $(\mathscr{S}_{n-m}, \widehat{\mathcal{A}}_\mathrm{s})$ are of positive type, whereas the remaining $k$ real eigenvalues $\mu_1^- \leq \dots \leq \mu_k^- < 0$ are of negative type. This corollary further implies that there exists some $\widehat{T} \in \mathrm{Gl}_{2(n-m)}$ with $\widehat{T}^\top \mathscr{S}_{n-m} \widehat{T} = \mathscr{S}_{n-m}$ and

$$\widehat{T}^{-1}\widehat{\mathcal{A}}_\mathrm{s}\widehat{T} = \begin{bmatrix} 0 & 0 & 0 & \mathcal{A}_{36} \\ 0 & \mathcal{A}_{44} & 0 & 0 \\ 0 & 0 & \mathcal{A}_{55} & 0 \\ -\mathcal{A}_{36}^\top & 0 & 0 & \mathcal{A}_{66} \end{bmatrix},$$

where, for some $\eta_i < 0$, $\nu_i, \in \mathbb{R}\backslash\{0\}$ for $i = 1, \ldots, c$, we have

$$\begin{aligned}
\mathcal{A}_{44} &= \operatorname{diag}(\mu_1^+, \ldots, \mu_k^+) \in \mathrm{Gl}_k(\mathbb{R}), \quad \mathcal{A}_{55} = \operatorname{diag}(\mu_1^-, \ldots, \mu_k^-) \in \mathrm{Gl}_k(\mathbb{R}),\\
\mathcal{A}_{66} &= \operatorname{diag}(\eta_1, \ldots, \eta_c) \in \mathrm{Gl}_c(\mathbb{R}), \qquad \mathcal{A}_{36} = \operatorname{diag}(\nu_1, \ldots, \nu_c) \in \mathrm{Gl}_k(\mathbb{R}).
\end{aligned}$$

Then $\widetilde{T} := \operatorname{diag}(I_{m+\ell}, \widehat{T}, I_{m+\ell})$ fulfills $\widetilde{T}^\top \mathscr{S}_n \widetilde{T} = \mathscr{S}_n$, and an application of this state space transformation leads to a realization $[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}]$ of the desired form. Further, since $\mathcal{A}_{18} = G_{31}^\top$, $\mathcal{A}_{27} = G_{22}^\top$, and $\mathcal{B}_8 = B_3$, those matrices are invertible by Lemma 5.1. Note that the statement that the upper left seven blocks of $\mathcal{A}_n$ correspond to the system zeros is also due to the latter lemma. In the remaining steps of the proof we show statement (d).

*Step* 2. Suppose that $P \geq 0$ solves $\mathscr{W}_{[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}, 0]}(P) \leq 0$ and partition $P = (P_{ij})_{i,j=1,\ldots,8}$ according to the block structure of $\mathcal{A}_\mathrm{n}$. We show that $P_{11} = P_{88} = I_m$ and $P_{1i} = 0$, $P_{j8} = 0$ for $i = 2, \ldots, 8$ and $j = 1, \ldots, 7$.

Suppose $P \geq 0$ solves the KYP inequality. Then the lower right block in the inequality $\mathscr{W}_{[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}, 0]}(P) \leq 0$ is zero. Thus the corresponding off-diagonal blocks have to be zero as well, which leads to $P\mathcal{B}_\mathrm{n} - \mathcal{C}_\mathrm{n}^\top = 0$. This together with $\mathcal{B}_8 \in \mathrm{Gl}_m(\mathbb{R})$ implies that $P_{j8} = 0$ for $j = 1, \ldots, 7$ and $P_{88} = I_m$. As a consequence the upper left block of size $m \times m$ of the left-hand side of $\mathcal{A}_\mathrm{n}^\top P + P\mathcal{A}_\mathrm{n} \leq 0$ is zero, hence all the corresponding off-diagonal blocks of $\mathcal{A}_\mathrm{n}^\top P + P\mathcal{A}_\mathrm{n}$ have to be zero as well. Equivalently, $-\mathcal{A}_{27}P_{17}^\top = 0$, $-\mathcal{A}_{36}P_{16}^\top = 0$, $\mathcal{A}_{44}P_{14}^\top = 0$, $\mathcal{A}_{55}P_{15}^\top = 0$, $\mathcal{A}_{36}^\top P_{13}^\top + \mathcal{A}_{66}P_{16}^\top = 0$, and $\mathcal{A}_{27}^\top P_{12}^\top = 0$. Since $\mathcal{A}_{27}$, $\mathcal{A}_{36}$, $\mathcal{A}_{44}$, $\mathcal{A}_{55}$, and $\mathcal{A}_{66}$ are invertible, $P_{1i} = 0$ for $i = 2, \ldots, 8$.

*Step* 3. We show that $P_{2i} = 0$ and $P_{7i} = 0$ for $i = 3, \ldots, 6$. We have that

$$\begin{bmatrix} 0 & -\mathcal{A}_{27} \\ \mathcal{A}_{27}^\top & 0 \end{bmatrix} \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} + \underbrace{\begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix}}_{=:Z} \underbrace{\begin{bmatrix} 0 & \mathcal{A}_{27} \\ -\mathcal{A}_{27}^\top & 0 \end{bmatrix}}_{=:Y} \leq 0, \tag{5.5}$$

where, due to Lemma 5.2, equality holds. We set

$$\breve{\mathcal{A}} := \begin{bmatrix} 0 & 0 & 0 & \mathcal{A}_{36} \\ 0 & \mathcal{A}_{44} & 0 & 0 \\ 0 & 0 & \mathcal{A}_{55} & 0 \\ -\mathcal{A}_{36}^\top & 0 & 0 & \mathcal{A}_{66} \end{bmatrix}, \quad \breve{P}_1 := \begin{bmatrix} P_{33} & \cdots & P_{36} \\ \vdots & & \vdots \\ P_{36}^\top & \cdots & P_{66} \end{bmatrix}, \quad \breve{P}_2 := \begin{bmatrix} P_{23}^\top & P_{37} \\ P_{24}^\top & P_{47} \\ P_{25}^\top & P_{57} \\ P_{26}^\top & P_{67} \end{bmatrix}.$$

By considering the principal submatrix of $\mathcal{A}_\mathrm{n}^\top P + P\mathcal{A}_\mathrm{n}$ obtained by removing the first and last $m$ rows, we obtain

$$0 \geq \begin{bmatrix} \breve{\mathcal{A}} & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} \breve{P}_1 & \breve{P}_2 \\ \breve{P}_2^\top & Z \end{bmatrix} + \begin{bmatrix} \breve{P}_1 & \breve{P}_2 \\ \breve{P}_2^\top & Z \end{bmatrix} \begin{bmatrix} \breve{\mathcal{A}}^\top & 0 \\ 0 & Y^\top \end{bmatrix} = \begin{bmatrix} \breve{\mathcal{A}}^\top \breve{P}_1 + \breve{P}_1 \breve{\mathcal{A}} & \breve{\mathcal{A}}\breve{P}_2 + \breve{P}_2 Y^\top \\ Y\breve{P}_2 + \breve{P}_2\breve{\mathcal{A}}^\top & 0 \end{bmatrix}$$

with $Z, Y$ as in (5.5). Thus, we obtain the Sylvester equation $Y\breve{P}_2^\top + \breve{P}_2^\top \breve{\mathcal{A}}^\top = 0$ for $\breve{P}_2$. The spectrum of $Y$ is contained on the imaginary axis, and all eigenvalues of $\breve{\mathcal{A}}^\top$ have negative real part. Since $Y$ and $-\breve{\mathcal{A}}^\top$ have no common eigenvalues, we obtain from [22, Thm. 2.4.4.1] that $\breve{P}_2 = 0$.

*Step* 4. We show that $\left(\left[\begin{smallmatrix} 0 & -\mathcal{A}_{27} \\ \mathcal{A}_{27}^\top & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \mathcal{A}_{28} \\ 0 \end{smallmatrix}\right]\right)$ is controllable. Assume that $\lambda \in \mathbb{C}$ and $v_2, v_7 \in \mathbb{C}^\ell$ such that

$$\begin{bmatrix} v_2^* & v_7^* \end{bmatrix} \begin{bmatrix} \lambda I_\ell & \mathcal{A}_{27} & -\mathcal{A}_{28} \\ -\mathcal{A}_{27}^\top & \lambda I_\ell & 0 \end{bmatrix} = 0.$$

Since $\left[\begin{smallmatrix} 0 & -\mathcal{A}_{27} \\ \mathcal{A}_{27}^\top & 0 \end{smallmatrix}\right]$ is skew-symmetric, it follows that $\lambda = \mathrm{i}\omega$ for some $\omega \in \mathbb{R}$. By setting $v := \left[\begin{smallmatrix} 0 & v_2^* & 0 & 0 & 0 & 0 & v_7^* & 0 \end{smallmatrix}\right]^* \in \mathbb{C}^{2n}$ with block structure according to that of $\mathcal{A}_\mathrm{n}$, we now obtain that $v^* \begin{bmatrix} \mathrm{i}\omega I_{2n} - \mathcal{A}_\mathrm{n} & \mathcal{B}_\mathrm{n} \end{bmatrix} = 0$. Since $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ is assumed to be stabilizable, $[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}]$ is stabilizable as well. This leads to $v = 0$, whence $v_2 = v_7 = 0$.

*Step* 5. Let $P \geq 0$, $Q \geq 0$ be solutions of the KYP inequalities $\mathscr{W}_{[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}, 0]}(P) \leq 0$ and $\mathscr{W}_{[\mathcal{A}_\mathrm{n}^\top, \mathcal{C}_\mathrm{n}^\top, \mathcal{B}_\mathrm{n}^\top, 0]}(Q) \leq 0$. We show that $P = \mathrm{diag}(I_{m+\ell}, P_2, I_{m+\ell})$ and $Q = \mathrm{diag}(I_{m+\ell}, Q_2, I_{m+\ell})$ for some $P_2, Q_2 \in \mathbb{R}^{2(c+k)\times 2(c+k)}$.

Since the solution $P \geq 0$ fulfills $\mathscr{W}_{[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}, 0]}(P) \leq 0$, if and only if $Q = \mathscr{S}_n P \mathscr{S}_n \geq 0$ fulfills $\mathscr{W}_{[\mathcal{A}_\mathrm{n}^\top, \mathcal{C}_\mathrm{n}^\top, \mathcal{B}_\mathrm{n}^\top, 0]}(Q) \leq 0$, it suffices to prove the statement only for $P \geq 0$ with $\mathscr{W}_{[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}, 0]}(P) \leq 0$. We partition $P = (P_{ij})_{i,j=1,\ldots,8}$ according to the block structure of $\mathcal{A}_\mathrm{n}$. Thus we use our findings in Step 2 and Step 3 to see that $P_{11} = P_{88} = I_m$ and $P_{1i} = 0$, $P_{j8} = 0$ for $i = 2, \ldots, 8$ and $j = 1, \ldots, 7$, and $P_{2i} = 0$ and $P_{7i} = 0$ for $i = 3, \ldots, 6$. Now a straightforward calculation yields that

(5.6)
$$\begin{bmatrix} 0 & -\mathcal{A}_{27} & -\mathcal{A}_{28} \\ \mathcal{A}_{27}^\top & 0 & 0 \\ \mathcal{A}_{28}^\top & 0 & \mathcal{A}_{88} \end{bmatrix} \begin{bmatrix} P_{22} & P_{27} & 0 \\ P_{27}^\top & P_{77} & 0 \\ 0 & 0 & I_m \end{bmatrix} + \begin{bmatrix} P_{22} & P_{27} & 0 \\ P_{27}^\top & P_{77} & 0 \\ 0 & 0 & I_m \end{bmatrix} \begin{bmatrix} 0 & \mathcal{A}_{27} & \mathcal{A}_{28} \\ -\mathcal{A}_{27}^\top & 0 & 0 \\ -\mathcal{A}_{28}^\top & 0 & \mathcal{A}_{88} \end{bmatrix} \leq 0.$$

An evaluation of the upper right two block gives

$$\begin{bmatrix} 0 & -\mathcal{A}_{27} \\ \mathcal{A}_{27}^\top & 0 \end{bmatrix} \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} + \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} \begin{bmatrix} 0 & \mathcal{A}_{27} \\ -\mathcal{A}_{27}^\top & 0 \end{bmatrix} \leq 0,$$

whence, by Lemma 5.2, the latter inequality becomes an equality. Invoking this, an evaluation of the blocks "31" and "32" leads to

$$\begin{bmatrix} \mathcal{A}_{28}^\top & 0 \end{bmatrix} \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{28}^\top & 0 \end{bmatrix}.$$

This altogether yields

$$\begin{bmatrix} 0 & \mathcal{A}_{27} \\ -\mathcal{A}_{27}^\top & 0 \end{bmatrix} \left( I_{2\ell} - \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} \right) = \left( I_{2\ell} - \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} \right) \begin{bmatrix} 0 & \mathcal{A}_{27} \\ -\mathcal{A}_{27}^\top & 0 \end{bmatrix},$$
$$\begin{bmatrix} \mathcal{A}_{28}^\top & 0 \end{bmatrix} \left( I_{2\ell} - \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} \right) = 0.$$

Hence, $\mathrm{im}(I_{2\ell} - \left[\begin{smallmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{smallmatrix}\right])$ is an $\left[\begin{smallmatrix} 0 & \mathcal{A}_{27} \\ -\mathcal{A}_{27}^\top & 0 \end{smallmatrix}\right]$-invariant subspace and is contained in $\ker \begin{bmatrix} \mathcal{A}_{28}^\top & 0 \end{bmatrix}$. However, by Step 4, $(\left[\begin{smallmatrix} 0 & -\mathcal{A}_{27} \\ \mathcal{A}_{27}^\top & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \mathcal{A}_{28} \\ 0 \end{smallmatrix}\right])$ is controllable, thus

$$\mathrm{im}\left( I_{2\ell} - \begin{bmatrix} P_{22} & P_{27} \\ P_{27}^\top & P_{77} \end{bmatrix} \right) = \{0\}$$

and $P_{22} = P_{77} = I_\ell$ and $P_{27} = 0$. $\square$

With the previous theorem at hand and the normal form therein we can now exploit the structure of the original system and the solutions of the KYP inequalities to find a positive real balanced realization of the original system of the form (1.4).

THEOREM 5.4. *Let a system $[\mathcal{A}, \mathcal{B}, \mathcal{C}] \in \Xi_{n,m}$ be given with $D \geq 0$ and transfer function $\mathbf{G}(s)$. Suppose that the system has semisimple zeros. Then $\mathbf{G}(s)$ has a positive real balanced realization $[\mathcal{A}_\mathrm{b}, \mathcal{B}_\mathrm{b}, \mathcal{C}_\mathrm{b}]$ of the block form*

$$\mathcal{A}_\mathrm{b} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \widehat{\mathcal{A}}_{16} \\ 0 & 0 & 0 & 0 & \widehat{\mathcal{A}}_{25} & \widehat{\mathcal{A}}_{26} \\ 0 & 0 & \widehat{\mathcal{A}}_{33} & \widehat{\mathcal{A}}_{34} & 0 & \widehat{\mathcal{A}}_{36} \\ 0 & 0 & -\widehat{\mathcal{A}}_{34}^\top & \widehat{\mathcal{A}}_{44} & 0 & \widehat{\mathcal{A}}_{46} \\ 0 & -\widehat{\mathcal{A}}_{25}^\top & 0 & 0 & 0 & 0 \\ -\widehat{\mathcal{A}}_{16}^\top & -\widehat{\mathcal{A}}_{26}^\top & -\widehat{\mathcal{A}}_{36}^\top & \widehat{\mathcal{A}}_{46}^\top & 0 & \widehat{\mathcal{A}}_{66} \end{bmatrix}, \ \mathcal{B}_\mathrm{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \widehat{\mathcal{B}}_6 \end{bmatrix} = \mathcal{C}_\mathrm{b}^\top, \tag{5.7}$$

*where* $\widehat{\mathcal{A}}_{16}, \widehat{\mathcal{B}}_6 \in \mathrm{Gl}_m(\mathbb{R})$, $\widehat{\mathcal{A}}_{66} \in \mathbb{R}^{m\times m}$, $\widehat{\mathcal{A}}_{33} \in \mathbb{R}^{p_1\times p_1}$, $\widehat{\mathcal{A}}_{44} \in \mathbb{R}^{p_2\times p_2}$, $\widehat{\mathcal{A}}_{25} \in \mathbb{R}^{\ell\times\ell}$ *and* $\widehat{n} = 2m+p_1+p_2+2\ell$ *is the order of the minimal system. Further, the minimal solutions of the KYP inequalities* $\mathscr{W}_{[\mathcal{A}_\mathrm{b},\mathcal{B}_\mathrm{b},\mathcal{C}_\mathrm{b},0]}(P) \le 0$ *and* $\mathscr{W}_{[\mathcal{A}_\mathrm{b}^\top,\mathcal{C}_\mathrm{b}^\top,\mathcal{B}_\mathrm{b}^\top,0]}(Q) \le 0$ *are given by* $P = Q = \mathrm{diag}(I_{m+\ell}, \Pi, I_{m+\ell})$ *for some diagonal matrix* $\Pi \in \mathbb{R}^{(p_1+p_2)\times(p_1+p_2)}$. *Moreover, the system*

$$\left[\begin{bmatrix} \widehat{\mathcal{A}}_{33} & \widehat{\mathcal{A}}_{34} \\ -\widehat{\mathcal{A}}_{34}^\top & \widehat{\mathcal{A}}_{44} \end{bmatrix}, \begin{bmatrix} \widehat{\mathcal{A}}_{36} \\ \widehat{\mathcal{A}}_{46} \end{bmatrix}, \begin{bmatrix} \widehat{\mathcal{A}}_{36}^\top & -\widehat{\mathcal{A}}_{46}^\top \end{bmatrix}, -\widehat{\mathcal{A}}_{66}\right] =: [\mathcal{A}_\mathrm{z}, \mathcal{B}_\mathrm{z}, \mathcal{C}_\mathrm{z}, \mathcal{D}_\mathrm{z}] \tag{5.8}$$

*is asymptotically stable and positive real balanced, where* $\Pi$ *is the minimal solution of* $\mathscr{W}_{[\mathcal{A}_\mathrm{z},\mathcal{B}_\mathrm{z},\mathcal{C}_\mathrm{z},\mathcal{D}_\mathrm{z}]}(\widehat{P}) \le 0$ *and* $\mathscr{W}_{[\mathcal{A}_\mathrm{z}^\top,\mathcal{C}_\mathrm{z}^\top,\mathcal{B}_\mathrm{z}^\top,\mathcal{D}_\mathrm{z}^\top]}(\widehat{Q}) \le 0$ *and the spectrum of* $\mathcal{A}_\mathrm{z}$ *coincides with the set of zeros of* $[\mathcal{A}_\mathrm{b}, \mathcal{B}_\mathrm{b}, \mathcal{C}_\mathrm{b}]$ *with negative real part.*

*Proof.* Theorem 5.3 provides that $\mathbf{G}(s)$ has a realization $[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}]$ of the block form (5.4), where $P_{\min}$ and $Q_{\min}$ have the block structure $P_{\min} = \mathrm{diag}(I_{m+\ell}, P_2, I_{\ell+m})$ and $Q_{\min} = \mathrm{diag}(I_{m+\ell}, Q_2, I_{\ell+m})$ for some matrices $P_2, Q_2 \le 0$. To derive a balanced and therefore minimal system we use the reduction of section 3, but solely truncate those states corresponding to zero diagonal elements of $\Sigma$ in (3.2). In detail, let $[\mathcal{A}_\mathrm{b}, \mathcal{B}_\mathrm{b}, \mathcal{C}_\mathrm{b}]$ be the reduced system of $[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}]$ constructed as in (3.7) for the choice $r^- = n - \dim\ker\Sigma^-$ and $r^+ = n - \dim\ker\Sigma^+$. Here, the matrices $W^\top$ and $V$ from (3.7) have the same block structure as $P_{\min}$, namely $W^\top = \mathrm{diag}(I_{m+\ell}, W_2^\top, I_{m+\ell})$ and $V = \mathrm{diag}(I_{m+\ell}, V_2, I_{m+\ell})$ for some matrices $W_2, V_2 \in \mathbb{R}^{(p_1+p_2)\times(p_1+p_2)}$. Inspecting the proof of Theorem 3.3, its statements are also valid for the reduced system $[\mathcal{A}_\mathrm{b}, \mathcal{B}_\mathrm{b}, \mathcal{C}_\mathrm{b}]$. Thus, by Theorem 3.3(b) and (d) it follows that $[\mathcal{A}_\mathrm{b}, \mathcal{B}_\mathrm{b}, \mathcal{C}_\mathrm{b}]$ is a minimal, positive real balanced realization of $\mathbf{G}(s)$ and

$$\mathrm{diag}(-I_{r^-}, I_{r^+})\mathcal{A}_\mathrm{b}\,\mathrm{diag}(-I_{r^-}, I_{r^+}) = \mathcal{A}_\mathrm{b}^\top.$$

Altogether, the realization $[\mathcal{A}_\mathrm{b}, \mathcal{B}_\mathrm{b}, \mathcal{C}_\mathrm{b}]$ has the block form (5.7). Moreover, Theorem 5.3 and the block form of the reduction matrices $V$ and $W$ provide that all solutions of the KYP inequalities $\mathscr{W}_{[\mathcal{A}_\mathrm{b},\mathcal{B}_\mathrm{b},\mathcal{C}_\mathrm{b},0]}(P) \le 0$ and $\mathscr{W}_{[\mathcal{A}_\mathrm{b}^\top,\mathcal{C}_\mathrm{b}^\top,\mathcal{B}_\mathrm{b}^\top,0]}(Q) \le 0$ have the block form $P = \mathrm{diag}(I_{m+\ell}, P_2, I_{m+\ell})$ and $Q = \mathrm{diag}(I_{m+\ell}, Q_2, I_{m+\ell})$ for some $P_2, Q_2 \in \mathbb{R}^{(p_1+p_2)\times(p_1+p_2)}$. This in particular holds for the minimal solutions of the two KYP inequalities, which we therefore write as $\mathrm{diag}(I_{m+\ell}, \Pi, I_{m+\ell})$ for some suitable diagonal matrix $\Pi \in \mathbb{R}^{(p_1+p_2)\times(p_1+p_2)}$.

Straightforward calculations show that the matrix $\Pi$ solves $\mathscr{W}_{[\mathcal{A}_\mathrm{z},\mathcal{B}_\mathrm{z},\mathcal{C}_\mathrm{z},\mathcal{D}_\mathrm{z}]}(\widehat{P}) \le 0$ and $\mathscr{W}_{[\mathcal{A}_\mathrm{z}^\top,\mathcal{C}_\mathrm{z}^\top,\mathcal{B}_\mathrm{z}^\top,\mathcal{D}_\mathrm{z}^\top]}(\widehat{Q}) \le 0$. Now, the minimality of $\Pi$ follows directly from the minimality of $\mathrm{diag}(I_{m+\ell}, \Pi, I_{m+\ell})$.

Recall that the zeros of $[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}]$ with negative real part are given by the union of the eigenvalues of $\mathcal{A}_{44}, \mathcal{A}_{55}$ and $\left[\begin{smallmatrix} 0 & \mathcal{A}_{36} \\ -\mathcal{A}_{36}^\top & \mathcal{A}_{66} \end{smallmatrix}\right]$ from (5.4). As positive real balanced realizations are minimal, the uncontrollable and unobservable modes of $[\mathcal{A}_\mathrm{n}, \mathcal{B}_\mathrm{n}, \mathcal{C}_\mathrm{n}]$ are removed by the above construction of $[\mathcal{A}_\mathrm{b}, \mathcal{B}_\mathrm{b}, \mathcal{C}_\mathrm{b}]$, while the remaining ones are preserved. The latter are, however, the eigenvalues of $\mathcal{A}_\mathrm{z}$, which leads to the fact that the spectrum of $\mathcal{A}_\mathrm{z}$ is contained in the open left complex half plane. ∎

The following remark is devoted to positive real balanced truncation of systems $[\mathcal{A},\mathcal{B},\mathcal{C}] \in \Xi_{n,m}$ with $D \geq 0$ by using the approach as in (3.2)–(3.7).

*Remark* 5.5. Let a system $[\mathcal{A},\mathcal{B},\mathcal{C}] \in \Xi_{n,m}$ be given with $D \geq 0$. Suppose that the system has semisimple zeros. We apply positive real balanced truncation by using the approach as in (3.2)–(3.7). In particular, we assume that there exist $r^+, r^- \in \mathbb{N}$ that fulfill (3.2)–(3.4) and $r^+ = r^- =: r$ such that the positive real characteristic values $\sigma^{\pm}_{q^{\pm}+1}, \dots, \sigma^{\pm}_{h^{\pm}}$ are all strictly below one. Note that Theorem 5.3 implies that $r \geq \ell + m$, where $2\ell$ is the number of nonzero and purely imaginary zeros of $[\mathcal{A},\mathcal{B},\mathcal{C}]$ (counted with multiplicities).

A positive real reduced order system can be constructed directly from the realization $[\mathcal{A}_{\mathrm{b}},\mathcal{B}_{\mathrm{b}},\mathcal{C}_{\mathrm{b}}]$ in Theorem 5.4 by truncating certain rows and columns which belong to the third and fourth block rows of the matrices with block structure as in (5.7). More precisely, the reduced order model has a realization of the form

$$[T^\top \mathcal{A}_{\mathrm{b}} T, T^\top \mathcal{B}_{\mathrm{b}}, \mathcal{C}_{\mathrm{b}} T] \quad \text{for } T = \operatorname{diag}\left(\begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I_r \end{bmatrix}\right). \tag{5.9}$$

However, a direct application of Theorem 3.3 does not necessarily result in the system $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}] = [T^\top \mathcal{A}_{\mathrm{b}} T, T^\top \mathcal{B}_{\mathrm{b}}, \mathcal{C}_{\mathrm{b}} T]$, but rather in a system which is similar to $[T^\top \mathcal{A}_{\mathrm{b}} T, T^\top \mathcal{B}_{\mathrm{b}}, \mathcal{C}_{\mathrm{b}} T]$ under a state space transformation which preserves the property of the signature structure together with $\operatorname{diag}(\Sigma_1^-, \Sigma_1^+)$ being a solution of the KYP inequalities for $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}]$ and $[\widetilde{\mathcal{A}}^\top,\widetilde{\mathcal{C}}^\top,\widetilde{\mathcal{B}}^\top]$. Such transformations are of block-diagonal and orthogonal type, where the sizes of the orthogonal block correspond to the multiplicities of the respective positive real characteristic values. As a consequence, the reduced system can be represented by

$$[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}] = [\widetilde{T}^\top \mathcal{A}_{\mathrm{b}} \widetilde{T}, \widetilde{T}^\top \mathcal{B}_{\mathrm{b}}, \mathcal{C}_{\mathrm{b}} \widetilde{T}],$$

where for some orthogonal matrices $\widetilde{U}_j^{\pm} \in \mathbb{R}^{n_j^{\pm} \times n_j^{\pm}}$ for $j = 1, \dots, h^{\pm}$

$$\widetilde{T} = \operatorname{diag}(\widetilde{U}_1^-, \dots \widetilde{U}_{h^-}^-, \widetilde{U}_{h^+}^+, \dots, \widetilde{U}_1^+) \cdot \operatorname{diag}\left(\begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I_r \end{bmatrix}\right).$$

**6. Construction of second order realizations.** Now, we treat the second order realization problem. In particular, we prove a necessary condition on the zero sign characteristics of the reduced system for having a representation as a second order realization. An important tool is the so-called standard triples [16, 17, 18, 25] for quadratic, i.e., degree two, matrix polynomials.

DEFINITION 6.1. *A standard triple is defined as a triple* $(X, Z, Y) \in \mathbb{R}^{n \times 2n} \times \mathbb{R}^{2n \times 2n} \times \mathbb{R}^{2n \times n}$ *such that*

$$\begin{bmatrix} X \\ XZ \end{bmatrix} \in \mathrm{Gl}_{2n}(\mathbb{R}) \quad \text{and} \quad \begin{bmatrix} X \\ XZ \end{bmatrix} Y = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}$$

*for some nonsingular matrix* $M \in \mathbb{R}^{n \times n}$.

*Let* $\mathbf{L}(s) = s^2 M + sD + K$ *for some* $M \in \mathrm{Gl}_n(\mathbb{R})$ *and* $D, K \in \mathbb{R}^{n \times n}$. *Then* $\mathbf{L}(s)$ *is said to be* generated *by a standard triple* $(X, Z, Y)$ *if* $M = (XZY)^{-1}$ *and the equation*

$$MXZ^2 + DXZ + KX = 0 \tag{6.1}$$

*is fulfilled.*

For a quadratic matrix polynomial $\mathbf{L}(s) = s^2 M + sD + K$, the tuple $(X, Z)$ from a standard triple encodes its right spectral data. This can best be seen assuming that $Z$ is diagonalizable over $\mathbb{C}$ and in Jordan canonical form. Then, the equality (6.1) implies that the $i$th diagonal entry $\lambda_i \in \mathbb{C}$ of $Z$ together with the $i$th column $x_i \in \mathbb{C}^n$ of $X$ solves the quadratic eigenvalue problem $(\lambda_i^2 M + \lambda_i D + K)x_i = 0$. Moreover, The tuple $(X, Z)$ from a standard triple together with a mass matrix $M$ completely determines a quadratic matrix polynomial with nonsingular leading term, as the next result shows.

THEOREM 6.2 (see [23, Thm. 1]). *A standard triple* $(X, Z, Y) \in \mathbb{R}^{n\times 2n} \times \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{2n\times n}$ *generates a uniquely defined quadratic matrix polynomial.*

Actually, in [23] standard triples are considered, where $Z$ is in Jordan canonical form, but this additional property is not made use of in the proof of the theorem above. Even though a quadratic matrix polynomial is uniquely defined by a standard triple, the reverse is not true in general. This can be seen as there is a one-to-one correspondence between standard triples of some quadratic matrix polynomial $\mathbf{L}(s)$ and realizations of $\mathbf{L}^{-1}(s)$.

THEOREM 6.3 (see [24, Thm. 14.2]). *Let* $\mathbf{L}(s) = s^2 M + sD + K$ *for some* $M \in \mathrm{Gl}_n(\mathbb{R})$ *and* $D, K \in \mathbb{R}^{n\times n}$. *Then,* $(X, Z, Y) \in \mathbb{R}^{n\times 2n} \times \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{2n\times n}$ *is a standard triple of* $\mathbf{L}(s)$ *if and only if* $\mathbf{L}^{-1}(s) = X(sI_{2n} - Z)^{-1}Y$. *Moreover,* $[Z, Y, X]$ *is minimal as a realization of* $\mathbf{L}^{-1}(s)$. *In particular, if* $K^{-1}$ *exists, then* $K^{-1} = -XZ^{-1}Y$.

It can be further inferred from Theorem 6.2 that the coefficients of $\mathbf{L}(s)$ are given by the so-called moments $\Gamma_j := XZ^jY$, i.e.,

$$M = \Gamma_1^{-1}, \quad D = -M\Gamma_2 M, \quad K = -M\Gamma_3 M + D\Gamma_1 D. \tag{6.2}$$

A standard triple $(X, Z, Y)$ is said to be *self-adjoint* if there exists a symmetric matrix $S \in \mathrm{Gl}_n(\mathbb{R})$ such that $SY = X^\top$, $Z^\top = SZS^{-1}$. The theorem above states that $[Z, Y, X]$ is the realization of a symmetric transfer function und thus $S$ is unique. Note that if $\mathbf{L}(s)$ possesses a self-adjoint standard triple, then the moments $\Gamma_j$ are symmetric and hence, by (6.2), the coefficients of $\mathbf{L}(s)$ are symmetric as well. Thus, by Theorem 6.3 together with (4.4), if one standard triple is self-adjoint, then all standard triples of $\mathbf{L}(s)$ are self-adjoint. Moreover, in this case the pole sign characteristics of $\mathbf{L}^{-1}(s)$ are given by the sign characteristics of $(S, Z)$ for any self-adjoint standard triple $(X, Z, S^{-1}X^\top)$ of $\mathbf{L}(s)$. We need to introduce some further notation.

DEFINITION 6.4. *Let* $(S, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ *be given, where* $S \in \mathrm{Gl}_n(\mathbb{R})$ *is symmetric and* $A$ *is* $S$*-self-adjoint and diagonalizable over* $\mathbb{C}$. *For* $\alpha < 0$, *we denote by* $n_-(\alpha)$ $(p_-(\alpha))$ *the number of eigenvalues of* $(S, A)$ *of negative (positive) type in* $[\alpha, 0)$ $((\alpha, 0])$ *and for* $\alpha > 0$, $n_+(\alpha)$ $(p_+(\alpha))$ *the number of eigenvalues of* $(S, A)$ *of negative (positive) type in* $[0, \alpha)$ $((0, \alpha])$.

Next we present a result from [26] that gives a necessary condition on the sign characteristics of quadratic matrix polynomials with positive definite leading and trailing coefficient.

THEOREM 6.5 (see [26, Thm. 16]). *Let* $(S, A) \in \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{2n\times 2n}$ *be given, where* $S = U^\top \mathscr{S}_n U$ *for some* $U \in \mathrm{Gl}_{2n}(\mathbb{R})$ *and* $A$ *is* $S$*-self-adjoint and diagonalizable over* $\mathbb{C}$. *Let* $\lambda_{\min}$ *and* $\lambda_{\max}$ *be its minimal and maximal real eigenvalue, respectively. Then there exists* $X \in \mathbb{R}^{n\times 2n}$ *such that* $X(sI_{2n} - A)^{-1}S^{-1}X^\top = (s^2M + sD + K)^{-1}$ *for some* $M > 0$, $K \geq 0$, *and* $D = D^\top$ *if and only if*

$$\begin{aligned} &n_-(\alpha) = p_-(\alpha) \quad \forall\, \alpha < \lambda_{\min} \quad \textit{and} \quad n_-(\alpha) \leq p_-(\alpha) \quad \forall\, \alpha \in [\lambda_{\min}, 0), \\ &n_+(\alpha) = p_+(\alpha) \quad \forall\, \alpha > \lambda_{\max} \quad \textit{and} \quad n_+(\alpha) \geq p_+(\alpha) \quad \forall\, \alpha \in (0, \lambda_{\max}]. \end{aligned} \tag{6.3}$$

*Remark* 6.6. In [26, Thm. 16], pairs $(S, A)$ are considered with a structure similar to the canonical form of Theorem 4.2. However, we have seen earlier that the sign characteristics of $(S, A)$ from any self-adjoint standard triple $(X, A, S^{-1}X^\top)$ of $\mathbf{L}(s)$ are exactly the pole sign characteristics of $\mathbf{L}^{-1}(s)$ and hence do not depend on the choice of the minimal realization of $\mathbf{L}^{-1}(s)$ or the standard triple of $\mathbf{L}(s)$.

We are now able to prove the main result of this section on a necessary and sufficient condition for our reduced system possessing a second order realization with positive definite leading and trailing coefficient. Note that the proof of this theorem is constructive, i.e., we can infer a method for constructing such a second order realization. We restrict ourselves to the case where the reduced system is minimal. However, later in section 8, when we present the whole reduction procedure, minimality is not required.

THEOREM 6.7. *Let a system $[\mathcal{A}, \mathcal{B}, \mathcal{C}] \in \Xi_{n,m}$ be given with $D \geq 0$. Suppose that the system has semisimple zeros. Let further $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ be the reduced system from Theorem* 3.3 *of order $2r$, where $r^+ = r^- =: r$. Suppose it is minimal and its zeros are semisimple. Then, the following are equivalent:*

(i) *The transfer function $\widetilde{\mathbf{G}}(s)$ of $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ has a second order realization of the form* (1.2), *where $\widetilde{K}, \widetilde{M} > 0$ and $\widetilde{D} = \widetilde{D}^\top \in \mathbb{R}^{r \times r}$ has at least $r - m$ positive eigenvalues.*

(ii) *The real zeros of $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ are given by $\mu_1^- \leq \cdots \leq \mu_k^-$ and $\mu_1^+ \leq \cdots \leq \mu_k^+ < \mu_{k+1}^+ = \cdots = \mu_{k+m}^+ = 0$ and fulfill $\mu_i^- < \mu_i^+$ for all $i = 1, \ldots, k$.*

(iii) *The transfer function $\widetilde{\mathbf{G}}(s)$ has a minimal realization $[\mathcal{A}_{\rm n}, \mathcal{B}_{\rm n}, \mathcal{C}_{\rm n}]$ structured as in* (5.4) *that fulfills the properties* (a)–(c) *from Theorem* 5.3 *and $\mathcal{A}_{44} - \mathcal{A}_{55} = \operatorname{diag}(\mu_1^+, \ldots, \mu_k^+) - \operatorname{diag}(\mu_1^-, \ldots, \mu_k^-) > 0$.*

*Proof.* (ii) $\Rightarrow$ (iii) Let $\mathbf{G}(s)$ be the transfer function of $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ and $[\widetilde{\mathcal{A}}_{\rm b}, \widetilde{\mathcal{B}}_{\rm b}, \widetilde{\mathcal{C}}_{\rm b}]$ be the positive real balanced realization given by Theorem 5.4 of the form (5.7). Due to Remark 5.5 we can assume w. l. o. g. that $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}] = [T^\top \mathcal{A}_{\rm b} T, T^\top \mathcal{B}_{\rm b}, \mathcal{C}_{\rm b} T]$ for $T = \operatorname{diag}\left(\left[\begin{smallmatrix} I_r \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ I_r \end{smallmatrix}\right]\right)$. Let

$$[\widetilde{\mathcal{A}}_{\rm z}, \widetilde{\mathcal{B}}_{\rm z}, \widetilde{\mathcal{C}}_{\rm z}, \mathcal{D}_{\rm z}] = [\widetilde{T}^\top \mathcal{A}_{\rm z} \widetilde{T}, \widetilde{T}^\top \mathcal{B}_{\rm z}, \mathcal{C}_{\rm z} \widetilde{T}, \mathcal{D}_{\rm z}]$$

for $\widetilde{T} := \operatorname{diag}\left(\left[\begin{smallmatrix} 0 \\ I_{r-m-\ell} \end{smallmatrix}\right], \left[\begin{smallmatrix} I_{r-m-\ell} \\ 0 \end{smallmatrix}\right]\right)$ and let $[\mathcal{A}_{\rm z}, \mathcal{B}_{\rm z}, \mathcal{C}_{\rm z}, \mathcal{D}_{\rm z}]$ be as in (5.8). Moreover, the eigenvalues of $\widetilde{\mathcal{A}}_{\rm z}$ are zeros of $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ and the real nonzero zeros of $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ are eigenvalues of $\widetilde{\mathcal{A}}_{\rm z}$. Since $[\widetilde{\mathcal{A}}_{\rm z}, \widetilde{\mathcal{B}}_{\rm z}, \widetilde{\mathcal{C}}_{\rm z}, \mathcal{D}_{\rm z}]$ is received by positive real balanced truncation of the minimal and asymptotically stable system $[\mathcal{A}_{\rm z}, \mathcal{B}_{\rm z}, \mathcal{C}_{\rm z}, \mathcal{D}_{\rm z}]$ from (5.8), the discussion in section III of [20] provides that the system $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ is asymptotically stable as well. Hence we can apply Corollary 4.3 to find some $\widehat{T} \in \mathrm{Gl}_{2p}(\mathbb{R})$ such that

$$\widehat{T}^{-1} \widetilde{\mathcal{A}}_{\rm z} \widehat{T} = \begin{bmatrix} 0 & 0 & \mathcal{V} \\ 0 & \Lambda & 0 \\ -\mathcal{V} & 0 & \mathcal{E} \end{bmatrix} \quad \text{and} \quad \widehat{T}^\top \mathscr{S}_p \widehat{T} = \mathscr{S}_p,$$

where $\Lambda = \operatorname{diag}(\mu_1^+, \ldots, \mu_k^+, \mu_1^-, \ldots, \mu_k^-)$ and $\mathcal{E} = \operatorname{diag}(\eta_1, \ldots, \eta_c)$ for $c := p - k$, $\mathcal{V} = \operatorname{diag}(\nu_1, \ldots, \nu_c)$, with $\eta_i < 0$ and $\nu_i \in \mathbb{R}$ for $i = 1, \ldots, c$. Applying the state space transformation $\breve{T} := \operatorname{diag}(I_{m+\ell}, \widehat{T}, I_{m+\ell})$ to $[\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$ then gives a minimal realization

$[\mathcal{A}_{\rm n},\mathcal{B}_{\rm n},\mathcal{C}_{\rm n}]$ of $\widetilde{\mathbf{G}}(s)$ structured as in Theorem 5.3 and which fulfills the properties (a)–(c) of this theorem. Now the zero sign characteristics of $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}]$ are given by the sign characteristics of $(\operatorname{diag}(I_m,-\mathscr{S}_{2k}),\operatorname{diag}(0,\Lambda))$. Hence, by assumption we get that $\operatorname{diag}(\mu_1^+,\ldots,\mu_k^+)-\operatorname{diag}(\mu_1^-,\ldots,\mu_k^-)>0$.

(iii) $\Rightarrow$ (ii) This follows since $\mu_1^+,\ldots,\mu_k^+$ and $\mu_1^-,\ldots,\mu_k^-$ are respectively the negative zeros of positive and negative type of the minimal realization $[\mathcal{A}_{\rm n},\mathcal{B}_{\rm n},\mathcal{C}_{\rm n}]$ of $\widetilde{\mathbf{G}}(s)$ which is structured as in (5.4) and $\widetilde{\mathbf{G}}(0)=\mathcal{C}_{\rm n}\mathcal{A}_{\rm n}^{-1}\mathcal{B}_{\rm n}=0$.

(iii) $\Rightarrow$ (i) Suppose we are given a minimal realization $[\mathcal{A}_{\rm n},\mathcal{B}_{\rm n},\mathcal{C}_{\rm n}]$ structured as in Theorem 5.3 and such that $\operatorname{diag}(\mu_1^+,\ldots,\mu_k^+)-\operatorname{diag}(\mu_1^-,\ldots,\mu_k^-)>0$. For $i=1,\ldots,k$ we set

$$a_i:=\sqrt{\tfrac{\mu_i^-}{\mu_i^--\mu_i^+}}\in\mathbb{R}\backslash\{0\},\quad b_i:=\sqrt{\tfrac{\mu_i^+}{\mu_i^--\mu_i^+}}\in\mathbb{R}\backslash\{0\},\quad T_i=\begin{bmatrix}a_i&b_i\\b_i&a_i\end{bmatrix}\in\mathrm{Gl}_2(\mathbb{R}).$$

Straightforward calculations give $T_i^{-1}=\mathscr{S}_2T_i\mathscr{S}_2$ and

$$T_i^{-1}\operatorname{diag}(\mu_i^+,\mu_i^-)T_i=\begin{bmatrix}0&*\\ *&\frac{(\mu_i^-)^2-(\mu_i^+)^2}{\mu_i^--\mu_i^+}\end{bmatrix}. \tag{6.4}$$

Note that since $\mu_i^\pm<0$ for all $i=1,\ldots,k$, the lower right entry of the matrix on the right-hand side is negative. We set $\widetilde{T}:=\left[\begin{smallmatrix}A&B\\B&A\end{smallmatrix}\right]$, where $A:=\operatorname{diag}(a_1,\ldots,a_k)$ and $B:=\operatorname{diag}(b_1,\ldots,b_k)$ and $T:=\operatorname{diag}(I_{r-k},\widetilde{T},I_{r-k})$. Interchanging some rows and columns, we arrive at a minimal realization $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}]$ of $\widetilde{\mathbf{G}}(s)$ of the form (1.5), where a second order realization can be derived directly with $\widetilde{M}=I_r$, $\widetilde{K}=\widetilde{G}\widetilde{G}^\top$, and $\widetilde{D}=\widetilde{D}^\top$. Note that $\widetilde{D}$ has the block form $\widetilde{D}=\left[\begin{smallmatrix}\widetilde{D}_{11}&\widetilde{D}_{12}\\\widetilde{D}_{12}^\top&\widetilde{D}_{22}\end{smallmatrix}\right]$, where $\widetilde{D}_{11}\in\mathbb{R}^{r-m\times r-m}$ and $\widetilde{D}_{22}\in\mathbb{R}^{m\times m}$ are positive semidefinite and diagonal matrices. Hence, Sylvester's law of inertia [22, Thm. 4.5.8] implies that $\widetilde{D}$ has at least $r-m$ nonnegative eigenvalues.

(i) $\Rightarrow$ (iii) Suppose $\widetilde{\mathbf{G}}(s)$ has a minimal second order realization of the form (1.2), where $\widetilde{M}\in\mathbb{R}^{r\times r}$. By Lemma 5.1, $\widetilde{\mathbf{G}}(s)$ then also has a minimal realization $[\mathcal{A}_{\rm s},\mathcal{B}_{\rm s},\mathcal{C}_{\rm s}]$ of the form (5.1), where its real and nonzero zeros coincide with the real eigenvalues of $\widehat{\mathcal{A}}_{\rm s}:=\left[\begin{smallmatrix}0&G_{13}^\top\\-G_{13}&-D_{11}\end{smallmatrix}\right]\in\mathrm{Gl}_{2p}(\mathbb{R})$ for $p:=r-m-\ell$ and the zero sign characteristics of $[\mathcal{A}_{\rm s},\mathcal{B}_{\rm s},\mathcal{C}_{\rm s}]$ coincide with the sign characteristics of $(\operatorname{diag}(I_m,-\mathscr{S}_p),\operatorname{diag}(0,\widehat{\mathcal{A}}_{\rm s}))$. Since, as mentioned in the first part, all zeros of $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}]$ have nonpositive real part, we can apply Theorem 5.3 to obtain a minimal realization $[\mathcal{A}_{\rm n},\mathcal{B}_{\rm n},\mathcal{C}_{\rm n}]$ of $\widetilde{\mathbf{G}}(s)$ of the form (5.4). Recall that the transformation $T\in\mathrm{Gl}_{2r}(\mathbb{R})$ we use to transform $[\mathcal{A}_{\rm s},\mathcal{B}_{\rm s},\mathcal{C}_{\rm s}]$ into $[\mathcal{A}_{\rm n},\mathcal{B}_{\rm n},\mathcal{C}_{\rm n}]$ has the block structure $T=\operatorname{diag}(I_{m+\ell},\widehat{T},I_{\ell+m})$, where $\widehat{T}\in\mathrm{Gl}_{2p}(\mathbb{R})$ has to fulfill (4.5), i.e., $\widehat{T}^\top\mathscr{S}_p\widehat{T}=\mathscr{S}_p$. This implies that the sign characteristics of $(-\mathscr{S}_p,\widehat{\mathcal{A}}_{\rm s})$ coincide with the sign characteristics of $(-\mathscr{S}_p,\operatorname{diag}(\mathcal{A}_{44},\mathcal{A}_{55}))$, with $\mathcal{A}_{44},\mathcal{A}_{55}$ from (5.4). Setting $X:=\begin{bmatrix}0&I_p\end{bmatrix}$ we obtain

$$\begin{bmatrix}X\widehat{\mathcal{A}}_{\rm s}^{-1}\\X\widehat{\mathcal{A}}_{\rm s}^{-1}\widehat{\mathcal{A}}_{\rm s}\end{bmatrix}=\begin{bmatrix}-G_{13}^{-\top}&0\\0&I_p\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}X\widehat{\mathcal{A}}_{\rm s}^{-1}\\X\widehat{\mathcal{A}}_{\rm s}^{-1}\widehat{\mathcal{A}}_{\rm s}\end{bmatrix}\mathscr{S}_pX^\top=\begin{bmatrix}0\\I_p\end{bmatrix}.$$

Thus, using (6.2) and Theorem 6.3, $(X\widehat{\mathcal{A}}_{\rm s}^{-1},\widehat{\mathcal{A}}_{\rm s},\mathscr{S}_pX^\top)$ is a standard triple for

$$\mu^2I_p-\mu X\widehat{\mathcal{A}}_{\rm s}\mathscr{S}_pX^\top-(X(\widehat{\mathcal{A}}_{\rm s}^{-1})^2\mathscr{S}_pX^\top)^{-1}=\mu^2I_p+\mu D_{11}+G_{13}G_{13}^\top.$$

Moreover, the standard triple is $S$-self-adjoint for $S := \mathscr{S}_p\widehat{\mathcal{A}}_{\mathrm{s}}^{-1}$, since

$$\begin{aligned} S\widehat{\mathcal{A}}_{\mathrm{s}}S^{-1} &= \mathscr{S}_p\widehat{\mathcal{A}}_{\mathrm{s}}^{-1}\widehat{\mathcal{A}}_{\mathrm{s}}\widehat{\mathcal{A}}_{\mathrm{s}}\mathscr{S}_p = \mathscr{S}_p\widehat{\mathcal{A}}_{\mathrm{s}}\mathscr{S}_p = \widehat{\mathcal{A}}_{\mathrm{s}}^{\top}, \\ (X\widehat{\mathcal{A}}_{\mathrm{s}}^{-1})^{\top} &= \widehat{\mathcal{A}}_{\mathrm{s}}^{-\top}X^{\top} = \widehat{\mathcal{A}}_{\mathrm{s}}^{-\top}\mathscr{S}_p\mathscr{S}_pX^{\top} = \mathscr{S}_p\widehat{\mathcal{A}}_{\mathrm{s}}^{-1}\mathscr{S}_pX^{\top} = S\mathscr{S}_pX^{\top}. \end{aligned}$$

Now Theorem 6.5 implies that for this standard triple, $n_-(\alpha) \le p_-(\alpha)$ for all $\alpha \in [\mu_{\min}, 0)$. Further, we get that

$$\widehat{T}^{\top}\widehat{\mathcal{A}}_{\mathrm{s}}^{-\top}\widehat{T}^{-\top} = \begin{bmatrix} -\mathcal{A}_{36}^{-\top}\mathcal{A}_{66}\mathcal{A}_{36}^{-1} & 0 & 0 & \mathcal{A}_{36}^{-\top} \\ 0 & \mathcal{A}_{44}^{-1} & 0 & 0 \\ 0 & 0 & \mathcal{A}_{55}^{-1} & 0 \\ -\mathcal{A}_{36}^{-1} & 0 & 0 & 0 \end{bmatrix}.$$

The sign characteristics only depend on the real eigenvalues and thus, $(\mathscr{S}_p\widehat{\mathcal{A}}_{\mathrm{s}}^{-1}, \widehat{\mathcal{A}}_{\mathrm{s}})$ and $(\widehat{T}^{\top}\mathscr{S}_p\widehat{T}\widehat{T}^{-1}\widehat{\mathcal{A}}_{\mathrm{s}}^{-1}\widehat{T}, \widehat{T}^{-1}\widehat{\mathcal{A}}_{\mathrm{s}}\widehat{T})$ have the same sign characteristics as the tuple $(\mathrm{diag}(\mathcal{A}_{44}^{-1}, \mathcal{A}_{55}^{-1})\mathscr{S}_p, \mathrm{diag}(\mathcal{A}_{44}, \mathcal{A}_{55}))$. Recall that $\mathcal{A}_{44}$ and $\mathcal{A}_{55}$ are negative definite. Hence, the sign characteristics of the latter coincide with the sign characteristics of $(-\mathscr{S}_p, \mathrm{diag}(\mathcal{A}_{44}, \mathcal{A}_{55}))$. Therefore, we get that $n_-(\alpha) \le p_-(\alpha)$ for all $\alpha \in [\mu_{\min}, 0)$ also holds for $(-\mathscr{S}_p, \mathrm{diag}(\mathcal{A}_{44}, \mathcal{A}_{55}))$, which means that $\mathcal{A}_{55} < \mathcal{A}_{44}$. ◻

We close this section with a discussion on the treatment of the necessary condition that is needed in order to recover a second order realization which we have learned previously.

*Remark* 6.8 (handling of the necessary condition). Unfortunately, the condition $\mathrm{diag}(\mu_1^+, \dots, \mu_k^+) - \mathrm{diag}(\mu_1^-, \dots, \mu_k^-) > 0$ on the zeros of the reduced system is not fulfilled in general. Moreover, we cannot guarantee establishing this property by small perturbations, since the sign characteristics are stable in the following sense. Consider a pair $(H, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, where $H$ is symmetric and nonsingular and $A$ is $H$-self-adjoint and diagonalizable over $\mathbb{C}$. Then for every simple real eigenvalue $\lambda$, there exist structure preserving neighborhoods $U_A$ of $A$, $U_H$ of $H$, and $U_\lambda$ of $\lambda$ such that if $A_1 \in U_A$ is $H_1$-self-adjoint for some $H_1 \in U_H$, then there is exactly one real simple eigenvalue $\lambda_1$ of $A_1$ in $U_\lambda$ and it holds that it has the same sign as $\lambda$; see [17, sect. III.5.1, Thm. 5.1]. Due to these facts, we have decided that whenever this condition fails, we add a certain subsystem in our reduction method after the truncation in order to recover this property instead of relying on perturbations of the system matrices; see section 8.

**7. Positive real balanced truncation for overdamped systems—the exceptional case.** Here we consider second order systems of the form (1.1) that additionally fulfill that $M$, $D$, and $K$ are all symmetric and positive definite, and the so-called overdamping condition

$$(v^*Dv)^2 > 4(v^*Mv)(v^*Kv) \;\forall\; v \in \mathbb{C}^n \tag{7.1}$$

is fulfilled. It is known that the overdamping condition implies that all zeros of $\det(s^2M + sD + K)$ are real (see [11]); it therefore generalizes the discriminant condition to the matrix-valued case. By forming a self-adjoint standard triple $(X, A, S^{-1}X^{\top})$ of $\mathbf{L}(s) = s^2M + sD + K$, we can define the sign characteristics of $\mathbf{L}(s)$ to be those of $(S, A)$ [15]. The sign characteristics of overdamped systems have a certain structure, which is summarized in the following result.

THEOREM 7.1 (see [1, Thm. 3.6]). *Let $(X, A, S^{-1}X^\top)$ form a self-adjoint standard triple of some matrix polynomial $s^2M + sD + K$, where $M, D, K \in \mathbb{R}^{n\times n}$ are symmetric. Then the following are equivalent:*

(i) *All eigenvalues of $(S, A)$ are real, semisimple, and negative and every eigenvalue of negative type is smaller than every eigenvalue of positive type. In detail, they fulfill $\lambda_1^- \leq \cdots \leq \lambda_n^- < \lambda_1^+ \leq \cdots \leq \lambda_n^+ < 0$.*

(ii) *It holds that $M, D, K > 0$ and (7.1).*

Noticing that the distribution of the real eigenvalues and their signs are system invariants, we obtain the following result.

COROLLARY 7.2. *Let a second order system be given of the form (1.1), with $M, D, K > 0$, which fulfills the condition (7.1), and let $\mathbf{G}(s)$ be its transfer function. Then every second order realization of the form (1.1) with symmetric $\widetilde{M}, \widetilde{D}, \widetilde{K} \in \mathbb{R}^{n\times n}$ fulfills $\widetilde{M}, \widetilde{D}, \widetilde{K} > 0$ and (7.1). In this case, for the first order representation $[\mathcal{A}, \mathcal{B}, \mathcal{C}]$ of the form (1.3), the eigenvalues of $(\mathscr{S}_n, \mathcal{A})$ fulfill $\lambda_1^+ \leq \cdots \leq \lambda_n^+ < \lambda_1^- \leq \cdots \leq \lambda_n^- < 0$.*

*Proof.* As in the last step of the proof of Theorem 6.7 one can show that the sign characteristics of $s^2M + sD + K$ are given by those of $(-\mathscr{S}_n, \mathcal{A})$. The second statement thus follows from Theorem 7.1. ∎

In order to show that positive real balanced truncation preserves the overdamping structure, we use some results of [8] and therefore need some notation.

DEFINITION 7.3. *Let $(S, A) \in \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{2n\times 2n}$, where $S$ is symmetric and nonsingular, and $A$ is $S$-self-adjoint and diagonalizable over $\mathbb{C}$. Consider the set $C^\pm := \{x \in \mathbb{C}^{2n} \mid x^*Sx \gtrless 0\}$ and, for a subspace $\mathcal{S} \subseteq \mathbb{C}^{2n}$, let*

(7.2)
$$\iota^+(\mathcal{S}) := \begin{cases} +\infty & \text{if } \mathcal{S} \cap C^+ = \emptyset, \\ \sup\limits_{x\in\mathcal{S}\cap C^+} \rho(x) & \text{otherwise}, \end{cases} \quad \iota^-(\mathcal{S}) := \begin{cases} -\infty & \text{if } \mathcal{S} \cap C^- = \emptyset, \\ \inf\limits_{x\in\mathcal{S}\cap C^-} \rho(x) & \text{otherwise}, \end{cases}$$

*where $\rho(x) := \frac{x^*SAx}{x^*Sx}$. We denote by*

$$\delta_h^+(S, A) := \{\inf \iota^+(\mathcal{S}) \mid \mathcal{S} \text{ is a subspace of } \mathbb{C}^{2n} \text{ with } \dim \mathcal{S} = 2n - h + 1\}$$
$$\text{and} \quad \sigma_h^-(S, A) := \{\sup \iota^-(\mathcal{S}) \mid \mathcal{S} \text{ is a subspace of } \mathbb{C}^{2n} \text{ with } \dim \mathcal{S} = 2n - h + 1\}.$$

Note that one has $\delta_{2n}^+(S, A) \leq \cdots \leq \delta_1^+(S, A)$ and $\sigma_1^-(S, A) \leq \cdots \leq \sigma_{2n}^-(S, A)$. We present two results from [8]. The second one treats the assignment of the values $\sigma_j^-(S, A)$ and $\delta_j^+(S, A)$ to eigenvalues of positive and negative type for tuples $(S, A)$ as in Definition 7.3 where all eigenvalues are real and $\lambda_1^+ \leq \cdots \leq \lambda_n^+ \leq \lambda_1^- \leq \cdots \leq \lambda_n^-$. In the first result, we study the values $\sigma_j^-(S, A)$ for tuples $(S, A)$ that do not satisfy such a distribution of the sign characteristics.

LEMMA 7.4 (see [8, Lem. 3.2]). *Let $(S, A) \in \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{2n\times 2n}$, where $S = U^\top \mathscr{S}_n U$ for some $U \in \mathrm{Gl}_{2n}(\mathbb{R})$ and $A$ is $S$-self-adjoint and diagonalizable over $\mathbb{C}$. Suppose the eigenvalues of $(S, A)$ do not satisfy $\lambda_1^+ \leq \cdots \leq \lambda_n^+ \leq \lambda_1^- \leq \cdots \leq \lambda_n^-$, i.e., $(S, A)$ has at least one nonreal eigenvalue, or there exists an eigenvalue of positive type which is larger than an eigenvalue of negative type. Then, $\sigma_{n+j}^+(S, A) = \infty$ for $1 \leq j \leq n$.*

THEOREM 7.5 (see [8, Cor. 4.4 and Thm. 4.3]). *Let $(S, A) \in \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{2n\times 2n}$, where $S = U^\top \mathscr{S}_n U$ for some $U \in \mathrm{Gl}_{2n}(\mathbb{R})$ and $A$ is $S$-self-adjoint and diagonalizable.*

*Suppose that all eigenvalues of* $(S,A)$ *are real and satisfy* $\lambda_1^+ \leq \cdots \leq \lambda_n^+ < \lambda_1^- \leq \cdots \leq \lambda_n^-$. *Then, for* $1 \leq j \leq n$ *we have*

$$\sigma_j^+(S,A) = -\infty, \qquad \sigma_{n+j}^+(S,A) = \lambda_j^+,$$
$$\delta_{2n-j+1}^-(S,A) = \lambda_j^-, \qquad \text{and} \qquad \delta_j^-(S,A) = \infty.$$

A consequence of Theorem 7.5 is that for $S$-self-adjoint and diagonalizable $A$ with only real eigenvalues which additionally fulfill $\lambda_1^+ \leq \cdots \leq \lambda_n^+ < \lambda_1^- \leq \cdots \leq \lambda_n^-$, this property is preserved under the application of certain two-sided reduction matrices on the pair $(S,A)$.

Lemma 7.6. *Let* $(S,A) \in \mathbb{R}^{2n\times 2n} \times \mathbb{R}^{2n\times 2n}$, *where* $S = U^\top \mathscr{S}_n U$ *for some* $U \in \mathrm{Gl}_{2n}(\mathbb{R})$ *and* $A$ *is* $S$*-self-adjoint. Suppose that* $A$ *is diagonalizable over* $\mathbb{C}$ *and that* $(S,A)$ *has only real eigenvalues that satisfy* $\lambda_1^+ \leq \cdots \leq \lambda_n^+ < \lambda_1^- \leq \cdots \leq \lambda_n^-$. *Let* $W, V \in \mathbb{R}^{2n\times g}$ *be full rank matrices with* $g \leq 2n$ *and suppose that for a symmetric matrix* $H \in \mathbb{R}^{g\times g}$ *it holds that* $HW^\top = V^\top S$ *and* $W^\top V = I_g$. *If the eigenvalues of* $(H, W^\top AV)$ *are semisimple, then they are all real and, if we denote them by* $\widetilde{\lambda}_1^\pm \leq \cdots \leq \widetilde{\lambda}_{g^\pm}^\pm$, *where* $g^+ + g^- = g$, *they fulfill* $\widetilde{\lambda}_1^+ \leq \cdots \leq \widetilde{\lambda}_{g^+}^+ < \widetilde{\lambda}_1^- \leq \cdots \leq \widetilde{\lambda}_{g^-}^-$.

*Proof.* We set $B := W^\top AV$ and for $v \in \mathbb{C}^g$ we write

$$\widetilde{\rho}(v) := \frac{v^* V^\top SAVv}{v^* V^\top SVv} = \frac{v^* HBv}{v^* Hv}.$$

By the previous theorem we have that $\lambda_n^+ = \sigma_{2n}^+(S,A) = \sup\{\rho(x) \mid x \in C^+\}$. Then,

$$\begin{aligned}\sigma_g^+(H,B) = \sup\{\widetilde{\rho}(v) \mid v \in \mathbb{C}^g,\ v^*Hv > 0\} &= \sup\{\rho(x) \mid x \in \mathbb{C}^{2n},\ x \in C^+ \cap \operatorname{im} V\} \\ &\leq \sigma_{2n}^+(S,A) < \infty.\end{aligned}$$

Now, from Lemma 7.4 it follows that all eigenvalues of $(H,B)$ are real and $\widetilde{\lambda}_1^+ \leq \cdots \leq \widetilde{\lambda}_{g^+}^+ \leq \widetilde{\lambda}_1^- \leq \cdots \leq \widetilde{\lambda}_{g^-}^-$. Hence, it remains to show that $\widetilde{\lambda}_{g^+}^+ < \widetilde{\lambda}_1^-$. Note that

$$\begin{aligned}\delta_g^-(H,B) = \inf\{\widetilde{\rho}(v) \mid v \in \mathbb{C}^g,\ v^*Hv < 0\} &= \inf\{\rho(x) \mid x \in \mathbb{C}^{2n},\ x \in C^- \cap \operatorname{im} V\} \\ &\geq \inf\{\rho(x) \mid x \in C^-\} = \delta_{2n}^-(S,A).\end{aligned}$$

In particular, by Theorem 7.5 it holds that $\widetilde{\lambda}_{g^+}^+ = \widetilde{\sigma}_g^+(H,B) \leq \sigma_{2n}^+(S,A) = \lambda_n^+ < \lambda_1^- = \delta_{2n}^-(S,A) \leq \delta_g^-(H,B) = \widetilde{\lambda}_1^-$. ∎

For the main theorem of this section we combine the lemma above with the results from section 5. More precisely, we prove that the structure of an overdamped second order system is preserved applying the reduction ansatz from section 3.

Theorem 7.7. *Let a system* $[\mathcal{A},\mathcal{B},\mathcal{C}] \in \Xi_{n,m}$ *be given with* $D > 0$. *Let* $M := I_n$ *and* $K := GG^\top$ *and assume they fulfill* (7.1). *Let further* $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}]$ *be the reduced system from Theorem* 3.3 *of order* $2r$ *for* $r := r^+ = r^-$ *and let* $\widetilde{\mathbf{G}}(s)$ *be its transfer function. Suppose that the eigenvalues of* $\widetilde{\mathcal{A}}$ *and the zeros of* $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}]$ *are all semisimple. Then all eigenvalues of* $\widetilde{\mathcal{A}}$ *are real and fulfill* $\widetilde{\lambda}_1^+ \leq \cdots \leq \widetilde{\lambda}_r^+ < \widetilde{\lambda}_1^- \leq \cdots \leq \widetilde{\lambda}_r^-$. *In particular, every second order realization of* $\widetilde{\mathbf{G}}(s)$ *of the form* (1.1) *with symmetric* $\widetilde{M}, \widetilde{D}, \widetilde{K} \in \mathbb{R}^{r\times r}$ *fulfills* $\widetilde{M}, \widetilde{D}, \widetilde{K} > 0$ *and* (7.1). *Moreover,* $\widetilde{\mathbf{G}}(s)$ *possesses such a second order realization.*

*Proof.* A combination of Lemma 5.1 and Corollary 7.2 implies that we can w. l. o. g. assume that $[\mathcal{A},\mathcal{B},\mathcal{C}]$ is of the form (5.1) and all eigenvalues of $(\mathscr{S}_n,\mathcal{A})$ are real and negative, and they fulfill $\lambda_1^+\le\cdots\le\lambda_n^+<\lambda_1^-\le\cdots\le\lambda_n^-<0$. Since $D>0$, the second to last block rows and columns of size $\ell$ in (5.1), and therefore also the corresponding blocks of $[\mathcal{A}_{\rm b},\mathcal{B}_{\rm b},\mathcal{C}_{\rm b}]$, a positive real balanced realization of the form (5.7) of the original system, are absent. Recall that the latter realization exists by Theorem 5.4. Due to Remark 5.5 we can w. l. o. g. assume that $[\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{C}}]=[T^\top\mathcal{A}_{\rm b}T,T^\top\mathcal{B}_{\rm b},\mathcal{C}_{\rm b}T]$ for $T=\operatorname{diag}\left(\left[\begin{smallmatrix}I_r\\0\end{smallmatrix}\right],\left[\begin{smallmatrix}0\\I_r\end{smallmatrix}\right]\right)$. Let

$$[\widetilde{\mathcal{A}}_{\rm z},\widetilde{\mathcal{B}}_{\rm z},\widetilde{\mathcal{C}}_{\rm z},\mathcal{D}_{\rm z}]=[\widetilde{T}^\top\mathcal{A}_{\rm z}\widetilde{T},\widetilde{T}^\top\mathcal{B}_{\rm z},\mathcal{C}_{\rm z}\widetilde{T},\mathcal{D}_{\rm z}]$$

for $\widetilde{T}:=\operatorname{diag}\left(\left[\begin{smallmatrix}0\\I_{r-m}\end{smallmatrix}\right],\left[\begin{smallmatrix}I_{r-m}\\0\end{smallmatrix}\right]\right)$ and $[\mathcal{A}_{\rm z},\mathcal{B}_{\rm z},\mathcal{C}_{\rm z},\mathcal{D}_{\rm z}]$ as in (5.8). Since the eigenvalues of $\mathcal{A}_{\rm b}$ are a subset of the eigenvalues of $\mathcal{A}$ they are all real and the eigenvalues of negative type of $(\operatorname{diag}(-I_{m+p_1},I_{p_2+m}),\mathcal{A}_{\rm b})$ are all strictly smaller than the eigenvalues of positive type of $(\operatorname{diag}(-I_{m+p_1},I_{p_2+m}),\mathcal{A}_{\rm b})$. Now the tuple $(\operatorname{diag}(-I_{m+p_1},I_{p_2+m}),\mathcal{A}_{\rm b})$ and reduction matrices $V:=T$ and $W^\top:=T^\top$ fulfill the assumptions of Lemma 7.6 and hence, so does $(T^\top\operatorname{diag}(-I_{m+p_1},I_{p_2+m})T,T^\top\mathcal{A}_{\rm b}T)=(\mathscr{S}_r,\widetilde{\mathcal{A}})$ and $\widetilde{T}$. It follows that the eigenvalues of $(\mathscr{S}_r,\widetilde{\mathcal{A}})$ and of $(\mathscr{S}_{r-m},\widetilde{\mathcal{A}}_{\rm z})$ are all real and fulfill $\widetilde{\lambda}_1^+\le\cdots\le\widetilde{\lambda}_r^+<\widetilde{\lambda}_1^-\le\cdots\le\widetilde{\lambda}_r^-$ and $\widetilde{\mu}_1^+\le\cdots\le\widetilde{\mu}_{r-m}^+<\widetilde{\mu}_1^-\le\cdots\le\widetilde{\mu}_{r-m}^-$, respectively. On the other hand, the system $[\widetilde{\mathcal{A}}_{\rm z},\widetilde{\mathcal{B}}_{\rm z},\widetilde{\mathcal{C}}_{\rm z},\mathcal{D}_{\rm z}]$ is received by positive real balanced truncation of the asymptotically stable and positive real balanced (and thus, minimal) system $[\mathcal{A}_{\rm z},\mathcal{B}_{\rm z},\mathcal{C}_{\rm z},\mathcal{D}_{\rm z}]$. Now, the discussion in section III of [20] provides that the system $[\widetilde{\mathcal{A}}_{\rm z},\widetilde{\mathcal{B}}_{\rm z},\widetilde{\mathcal{C}}_{\rm z},\mathcal{D}_{\rm z}]$ is asymptotically stable as well and hence, $\widetilde{\mu}_{r-m}^+<0$. In particular, $\widetilde{\mathcal{A}}_{\rm z}$ is invertible. Since by Theorem 5.4, $\widehat{\mathcal{A}}_{16}\in\mathrm{Gl}_m(\mathbb{R})$, which is the block matrix on the upper right of $\mathcal{A}_{\rm b}$ and also of $\widetilde{\mathcal{A}}$, it follows from the block form of $\widetilde{\mathcal{A}}$ that it is invertible as well. By the passivity of the reduced system we have that $\widetilde{\lambda}_r^+\le 0$ and hence, $\widetilde{\lambda}_r^+<0$. We have now shown that property (a) from Theorem 7.1 is fulfilled for the pole sign characteristics of the reduced system.

Since the second condition of Theorem 6.7 is fulfilled, we follow the steps (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) to obtain a state space transformation $\breve{T}\in\mathrm{Gl}_{2r}(\mathbb{R})$ such that $\breve{T}^{-1}\widetilde{\mathcal{A}}\breve{T}$, $\breve{T}^{-1}\widetilde{\mathcal{B}}$, and $\widetilde{\mathcal{C}}\breve{T}$ are structured as in (3.1) for some matrices $\widetilde{G}\in\mathrm{Gl}_r(\mathbb{R})$, $\widetilde{B}\in\mathbb{R}^{r\times m}$, and some symmetric matrix $\widetilde{D}\in\mathbb{R}^{r\times r}$. Note that the minimality assumption of Theorem 6.7 is not needed in order to derive the transformation $\breve{T}$. This shows that $\widetilde{\mathbf{G}}(s)$ possesses a second order realization of the form (1.1) with symmetric coefficients $\widetilde{M}=I_r$, $\widetilde{K}=\widetilde{G}\widetilde{G}^\top$, and $\widetilde{D}$. Now Theorem 7.1 provides that it fulfills $\widetilde{M},\widetilde{D},\widetilde{K}>0$ and (7.1). The rest of the theorem follows from Corollary 7.2. ∎

**8. Numerical aspects.** This part is devoted to the numerical issues of the presented results. One problem that occurs considering the original large-scale system is that we need some factorization of the mass and stiffness matrices $M$ and $K$. Here, for many applications in mechanics those are band matrices or have an equally sparse structure, a property that one can be exploited to compute sparse Cholesky factorizations.

The bottleneck in the computation of the reduced order system is the determination of the minimal solution of the KYP inequality $\mathscr{W}_{[\mathcal{A},\mathcal{B},\mathcal{C},0]}(P)\le 0$. Here one can use the method from [31], which provides the minimal solution directly in factored form. We use this method and apply the Riccati iteration with the RADI algorithm [6] and with $L^\top DL$-factored intermediate iterates to solve the projected Riccati equations that are constructed in [31]. That is, this method delivers low rank approximations of the type $P_{\min} \approx L^\top L$ for a matrix $L$ which has a small number of rows compared to the number of columns. We arrive at the following numerical procedure.

**Numerical procedure: Second order positive real balanced truncation and structure recovery.** For a second order system of the form (1.1), where $M, K > 0$ and $D \geq 0$, compute a reduced system of the form (1.2), where $\widetilde{M}, \widetilde{K} > 0$ and $\widetilde{D} = \widetilde{D}^\top$.

**Inputs:** Matrices $M, K > 0$, $D \geq 0$, and $B$, a tolerance value **tol** $> 0$.

**Outputs:** Matrices $\widetilde{M}, \widetilde{K} > 0$, $\widetilde{D} = \widetilde{D}^\top$, and $\widetilde{B}$, estimates of the gap metric error bound **err** and a relative gap metric error bound **relerr**.

(1) Start with the reduction by positive real balanced truncation as presented in section 3:

(a) Use the sparse Cholesky factors $G$ and $H$ of $M, K$ to derive the first order representation (1.3).

(b) Compute a low rank factor $L \in \mathbb{R}^{k \times 2n}$ with $k \ll n$ such that $P_{\min} \approx L^\top L$. One can use, for example, the solver introduced in [31] that computes low rank factors of a matrix $P > 0$ from a *stabilizing solution triple* $(P, \mathcal{K}, \mathcal{L}) \in \mathbb{R}^{2n \times 2n} \times \mathbb{R}^{m \times 2n} \times \mathbb{R}^{m \times m}$ of the so-called Lur'e equation

$$\begin{bmatrix} \mathcal{A}^\top P + P\mathcal{A} & P\mathcal{B} - \mathcal{C}^\top \\ \mathcal{B}^\top P - \mathcal{C} & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{K}^\top \\ \mathcal{L}^\top \end{bmatrix} \begin{bmatrix} \mathcal{K} & \mathcal{L} \end{bmatrix} = 0, \tag{8.1}$$

where the term "stabilizing" refers to the rank condition

$$\operatorname{rank} \begin{bmatrix} -\lambda I_{2n} + \mathcal{A} & \mathcal{B} \\ \mathcal{K} & \mathcal{L} \end{bmatrix} = n + \operatorname{rank} \begin{bmatrix} \mathcal{K} & \mathcal{L} \end{bmatrix} \quad \forall\, \lambda \in \mathbb{C}^+.$$

(c) Take the partitioned eigendecomposition

$$L \mathscr{S}_n L^\top = \begin{bmatrix} U_1^- & U_2 & U_1^+ \end{bmatrix} \begin{bmatrix} -\Sigma_1^- & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & \Sigma_1^+ \end{bmatrix} \begin{bmatrix} (U_1^-)^\top \\ (U_2)^\top \\ (U_1^+)^\top \end{bmatrix}, \tag{8.2}$$

and define reduction matrices $W^\top := \Sigma_1^{-\frac{1}{2}} \mathscr{S}_r U_1^\top L$ and $V := \mathscr{S}_n L^\top U_1 \Sigma_1^{-\frac{1}{2}}$, where $\Sigma_1 := \operatorname{diag}(\Sigma_1^-, \Sigma_1^+)$ and $U_1 := \begin{bmatrix} U_1^- & U_1^+ \end{bmatrix}$, to compute a reduced first order model

$$\dot{\widetilde{x}}_r(t) = \widetilde{\mathcal{A}}_r \widetilde{x}_r(t) + \widetilde{\mathcal{B}}_r u(t), \quad \widetilde{y}_r(t) = \widetilde{\mathcal{C}}_r \widetilde{x}_r(t), \tag{8.3}$$

where $\widetilde{\mathcal{A}}_r := W^\top \mathcal{A} V = \left[\begin{smallmatrix} \widetilde{\mathcal{A}}_{r,11} & \widetilde{\mathcal{A}}_{r,12} \\ \widetilde{\mathcal{A}}_{r,21} & \widetilde{\mathcal{A}}_{r,22} \end{smallmatrix}\right]$ with $\widetilde{\mathcal{A}}_{r,ij} \in \mathbb{R}^{r \times r}$ for $i, j = 1, 2$, $\widetilde{\mathcal{B}}_r := W^\top \mathcal{B}$, and $\widetilde{\mathcal{C}}_r := \mathcal{C} V$. The gap metric error bound and a relative gap metric error bound can be *estimated* by

$$\mathbf{err} \approx 2 \sum_{i=1}^{k-2r} |\widetilde{\sigma}_i| + 2(2n-k)|\widetilde{\sigma}_{k-2r}|,$$
$$\mathbf{relerr} \approx \frac{\mathbf{err}}{\mathbf{err} + 2\sum_{i=1}^{r} (\widehat{\sigma}_i^+ + \widehat{\sigma}_i^-)},$$

where $\Sigma_2 = \operatorname{diag}(\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_{k-2r}) \in \mathbb{R}^{(k-2r)\times(k-2r)}$ with $|\widetilde{\sigma}_1| \geq \cdots \geq |\widetilde{\sigma}_{k-2r}|$ and $\Sigma_1^\pm = \operatorname{diag}(\widehat{\sigma}_1^\pm, \ldots, \widehat{\sigma}_r^\pm) \in \mathbb{R}^{r \times r}$.

(d) Due to the approximation error of computing the low rank factor $L$, the model (8.3) may not be internally passive. Restore the internal symmetry structure by setting $\widetilde{\mathcal{B}} = \widetilde{\mathcal{C}}^\top := \frac{1}{2}(\widetilde{\mathcal{B}}_r + \widetilde{\mathcal{C}}_r^\top)$ and

$$\widetilde{\mathcal{A}} := \begin{bmatrix} \frac{1}{2}(\widetilde{\mathcal{A}}_{r,11} + \widetilde{\mathcal{A}}_{r,11}^\top) - V_1\Lambda_1^+V_1^\top & \frac{1}{2}(\widetilde{\mathcal{A}}_{r,12} - \widetilde{\mathcal{A}}_{r,21}^\top) \\ \frac{1}{2}(-\widetilde{\mathcal{A}}_{r,12} + \widetilde{\mathcal{A}}_{r,21}^\top) & \frac{1}{2}(\widetilde{\mathcal{A}}_{r,22} + \widetilde{\mathcal{A}}_{r,22}^\top) - V_2\Lambda_2^+V_2^\top \end{bmatrix},$$

where $\Lambda_1^+$ and $\Lambda_2^+$ are diagonal matrices containing the positive eigenvalues of $\frac{1}{2}(\widetilde{\mathcal{A}}_{r,11} + \widetilde{\mathcal{A}}_{r,11}^\top)$ and $\frac{1}{2}(\widetilde{\mathcal{A}}_{r,22} + \widetilde{\mathcal{A}}_{r,22}^\top)$ on their diagonal, respectively, and where $V_1$ and $V_2$ are the corresponding eigenvector matrices with orthonormal columns. Optionally, the structural zero entries in $\widetilde{\mathcal{A}}$ can be set to zero. It is recommended to choose significantly more ADI iterations than the order $2r$ of the reduced first order model to decrease the *relative Frobenius norm structural error*

$$\varepsilon_{\text{struct}} := \frac{\sqrt{\|\widetilde{\mathcal{A}} - \widetilde{\mathcal{A}}_r\|_{\mathrm{F}}^2 + \|\widetilde{\mathcal{B}} - \widetilde{\mathcal{B}}_r\|_{\mathrm{F}}^2 + \|\widetilde{\mathcal{B}}^\top - \widetilde{\mathcal{C}}_r\|_{\mathrm{F}}^2}}{\sqrt{\|\widetilde{\mathcal{A}}_r\|_{\mathrm{F}}^2 + \|\widetilde{\mathcal{B}}_r\|_{\mathrm{F}}^2 + \|\widetilde{\mathcal{C}}_r\|_{\mathrm{F}}^2}}.$$

(2) Continue with the second order structure recovery from sections 5 and 6.

(a) If necessary, then apply a block orthogonal state space transformation as in Remark 5.5 to derive a realization $[\widetilde{\mathcal{A}}_{\mathrm{b}}, \widetilde{\mathcal{B}}_{\mathrm{b}}, \widetilde{\mathcal{C}}_{\mathrm{b}}]$ of the form (5.7) with, additionally, $p_1 = p_2 =: p$.

(b) For $\widetilde{\mathcal{A}}_{\mathrm{z}} := \left[\begin{smallmatrix} \widetilde{\mathcal{A}}_{33} & \widetilde{\mathcal{A}}_{34} \\ -\widetilde{\mathcal{A}}_{34}^\top & \widetilde{\mathcal{A}}_{44}\end{smallmatrix}\right] \in \mathbb{R}^{2p\times 2p}$, the principal submatrix of $[\widetilde{\mathcal{A}}_{\mathrm{b}}, \widetilde{\mathcal{B}}_{\mathrm{b}}, \widetilde{\mathcal{C}}_{\mathrm{b}}]$ partitioned as in (5.7), compute $\widetilde{T} \in \mathrm{Gl}_{2p}(\mathbb{R})$ such that

$$\widetilde{T}^{-1}\widetilde{\mathcal{A}}_{\mathrm{z}}\widetilde{T} = \operatorname{diag}(\breve{\mathcal{A}}_{22}, \breve{\mathcal{A}}_{33})$$

with matrices $\breve{\mathcal{A}}_{22} = \operatorname{diag}(\widetilde{\mu}_1, \ldots, \widetilde{\mu}_{2\widetilde{k}})$ and $\breve{\mathcal{A}}_{33} = \operatorname{diag}(\mathscr{P}_{\sigma_1,\tau_1}, \ldots, \mathscr{P}_{\sigma_c,\tau_c})$, where $\mathscr{P}_{\sigma_1,\tau_1} = \left[\begin{smallmatrix} \sigma_i & \tau_i \\ -\tau_i & \sigma_i\end{smallmatrix}\right]$. Perform a state space transformation with the matrix $\operatorname{diag}(I_{m+\ell}, \widetilde{T}, I_{m+\ell})$ to derive a realization $[\breve{\mathcal{A}}, \breve{\mathcal{B}}, \breve{\mathcal{C}}]$ that we partition as

$$\breve{\mathcal{A}} = \begin{bmatrix} 0 & 0 & 0 & \breve{\mathcal{A}}_{14} \\ 0 & \breve{\mathcal{A}}_{22} & 0 & \breve{\mathcal{A}}_{24} \\ 0 & 0 & \breve{\mathcal{A}}_{33} & \breve{\mathcal{A}}_{34} \\ -\breve{\mathcal{A}}_{14}^\top & \breve{\mathcal{A}}_{42} & \breve{\mathcal{A}}_{43} & \breve{\mathcal{A}}_{44} \end{bmatrix}, \quad \breve{\mathcal{A}}_{24} = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{2\widetilde{k}} \end{bmatrix}, \quad \breve{\mathcal{A}}_{42}^\top = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{2\widetilde{k}} \end{bmatrix}, \tag{8.4}$$
$$\widetilde{\mathcal{A}}_{34}^\top = \begin{bmatrix} \mathbf{c}_1 & \mathbf{d}_1 & \cdots & \mathbf{c}_c & \mathbf{d}_c \end{bmatrix}, \; \breve{\mathcal{A}}_{43} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{f}_1 & \cdots & \mathbf{e}_c & \mathbf{f}_c \end{bmatrix}.$$

(c) To retrieve the symmetry structure construct a transformation $\breve{T}$:

- For $i \in \{1, \ldots, q\}$ choose $j \in \{1, \ldots, m\}$ such that $|a_{i,j}|\cdot|b_{i,j}| > \mathbf{tol}$, where $a_{i,j}, b_{i,j}$ are the $j$th entries of $\mathbf{a}_i$ and $\mathbf{b}_i$, respectively, and set $\breve{t}_i := \left|\frac{a_{i,j}}{b_{i,j}}\right|$.
- For $i \in \{1, \ldots, c\}$ we either find some $j \in \{1, \ldots, m\}$ such that for the $j$th entries of $\mathbf{c}_i, \mathbf{d}_i, \mathbf{e}_i$, and $\mathbf{f}_i$, which we call $c_{i,j}, d_{i,j}, e_{i,j}, d_{i,j}$, respectively, $|c_{i,j}|, |f_{i,j}|, |e_{i,j}|, |d_{i,j}| > \mathbf{tol}$, or $\sigma_i + \mathrm{i}\tau_i$ is close to being an unobservable or uncontrollable mode in which case $|c_{i,j}|, |f_{i,j}|, |e_{i,j}|, |d_{i,j}| \le \mathbf{tol}$ for all $j \in \{1, \ldots, m\}$. In the first case set $z_i^2 := \frac{d_i^2 + c_i^2}{e_i^2 + f_i^2}$ and find $\left[\begin{smallmatrix} \widetilde{x}_i \\ \widetilde{y}_i \end{smallmatrix}\right] \neq 0$ which solves
$$\begin{bmatrix} e_i z_i^2 - d_i & -f_i z_i^2 - c_i \\ f_i z_i^2 - c_i & e_i z_i^2 + d_i \end{bmatrix} \begin{bmatrix} \widetilde{x}_i \\ \widetilde{y}_i \end{bmatrix} = 0.$$
Scale the vector above via $\left[\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right] := \sqrt{\frac{z_i}{\widetilde{x}_i^2 + \widetilde{y}_i^2}}\left[\begin{smallmatrix} \widetilde{x}_i \\ \widetilde{y}_i \end{smallmatrix}\right]$ and set $\breve{T}_i := \left[\begin{smallmatrix} x_i & y_i \\ -y_i & x_i \end{smallmatrix}\right]$.

- If for all $j \in \{1, \ldots, m\}$, $|a_{i,j}| \cdot |b_{i,j}| \leq \mathbf{tol}$ or $|c_{i,j}|, |f_{i,j}|, |e_{i,j}|, |d_{i,j}| \leq \mathbf{tol}$ we can freely choose $\breve{t}_i = 1$ or $\breve{T}_i = I_2$.

Define the transformation $\breve{T} := \operatorname{diag}(I_{m+\ell}, \breve{t}_1, \ldots, \breve{t}_{2\widetilde{k}}, \breve{T}_1, \ldots, \breve{T}_c, I_{m+\ell})$ to get a realization $[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}, \widehat{\mathcal{C}}] := [\breve{T}^{-1}\breve{\mathcal{A}}\breve{T}, \breve{T}^{-1}\breve{\mathcal{B}}, \breve{\mathcal{C}}\breve{T}]$. Apply a state space transformation $\breve{V} := \operatorname{diag}(I_{m+\ell+2\widetilde{k}}, \Theta_1, \ldots, \Theta_c, I_{m+\ell})$, where $\Theta_i$ is taken from (4.3) and exchange the rows and columns to get a realization $[\mathcal{A}_{\mathrm{n}}, \mathcal{B}_{\mathrm{n}}, \mathcal{C}_{\mathrm{n}}]$ of the form of (5.4).

*Note*. The fact that $z_i > 0$ and thus $\sqrt{z_i} \in \mathbb{R}$ is guaranteed by the following arguments: By following the steps in the proof of (ii) $\Rightarrow$ (iii) in Theorem 6.5, we see that there exists a state space transformation for the reduced system that leads to a system of the form $[\mathcal{A}_{\mathrm{n}}, \mathcal{B}_{\mathrm{n}}, \mathcal{C}_{\mathrm{n}}]$. Moreover, such a state space transformation, except from the rows and columns corresponding to unobservable or uncontrollable modes, has to be of the form $\breve{T}\breve{V}$.

(d) If the submatrices $\mathcal{A}_{44} = \operatorname{diag}(\widetilde{\mu}_1^+, \ldots, \widetilde{\mu}_{\widetilde{k}}^+)$ and $\mathcal{A}_{55} = \operatorname{diag}(\widetilde{\mu}_1^-, \ldots, \widetilde{\mu}_{\widetilde{k}}^-)$ from $\mathcal{A}_{\mathrm{n}}$ as in (5.4) do not fulfill $\mathcal{A}_{44} - \mathcal{A}_{55} > 0$, find

$$\begin{aligned} &0 > \widetilde{\mu}_q^+ \geq \cdots \geq \widetilde{\mu}_{\widetilde{k}+1}^+ > \max\{\widetilde{\mu}_{\widetilde{k}}^-, \widetilde{\mu}_{\widetilde{k}}^+\} \text{ and} \\ &\widetilde{\mu}_q^- \leq \cdots \leq \widetilde{\mu}_{\widetilde{k}+1}^- < \min\{\widetilde{\mu}_1^-, \widetilde{\mu}_1^+\} \end{aligned}$$

and replace $\mathcal{A}_{44}$, $\mathcal{A}_{55}$, $\mathcal{A}_{48}^\top$, $\mathcal{A}_{58}^\top$ with

$$\begin{aligned} \breve{\mathcal{A}}_{44} &:= \operatorname{diag}(\mathcal{A}_{44}, \Lambda^+), & \breve{\mathcal{A}}_{55} &:= \operatorname{diag}(\Lambda^-, \mathcal{A}_{55}), \\ \breve{\mathcal{A}}_{48}^\top &:= \begin{bmatrix} \mathcal{A}_{48}^\top & 0 \end{bmatrix}, & \breve{\mathcal{A}}_{58}^\top &:= \begin{bmatrix} 0 & \mathcal{A}_{58}^\top \end{bmatrix}, \end{aligned}$$

where $\Lambda^- = \operatorname{diag}(\widetilde{\mu}_q^-, \ldots, \widetilde{\mu}_{\widetilde{k}+1}^-)$ and $\Lambda^+ = \operatorname{diag}(\widetilde{\mu}_{\widetilde{k}+1}^+, \ldots, \widetilde{\mu}_q^+)$ such that $\breve{\mathcal{A}}_{44} - \breve{\mathcal{A}}_{55} > 0$. Abusing notation we rename $\breve{\mathcal{A}}_{55} = \operatorname{diag}(\widetilde{\mu}_1^-, \ldots, \widetilde{\mu}_q^-)$.

(e) For $i = 1, \ldots, q$ set $a_i := \sqrt{\frac{\widetilde{\mu}_i^-}{\widetilde{\mu}_i^- - \widetilde{\mu}_i^+}}$, $b_i := \sqrt{\frac{\widetilde{\mu}_i^+}{\widetilde{\mu}_i^- - \widetilde{\mu}_i^+}}$, and

$$T_i := \begin{bmatrix} a_i & b_i \\ b_i & a_i \end{bmatrix}, \quad T := \operatorname{diag}(I_{m+\ell}, T_1, \ldots, T_{2\widehat{k}}, I_{\ell+m}).$$

Apply the state space transformation $T$ to the system.

(f) Suitable block exchanges brings the system into a realization $[\mathcal{A}_{\mathrm{s}}, \mathcal{B}_{\mathrm{s}}, \mathcal{C}_{\mathrm{s}}]$ of the block form (5.1) from Lemma 5.1. A second order realization of $\widetilde{\mathbf{G}}(s)$ is given by

$$\begin{aligned} \ddot{\widetilde{\xi}}(t) + \underbrace{\begin{bmatrix} D_{11} & D_{12} \\ D_{12}^\top & D_{22} \end{bmatrix}}_{=:\widetilde{D}} \dot{\widetilde{\xi}}(t) + \underbrace{\begin{bmatrix} 0 & G_{21}^\top \\ G_{12}^\top & G_{22}^\top \end{bmatrix} \begin{bmatrix} 0 & G_{12} \\ G_{21} & G_{22} \end{bmatrix}}_{=:\widetilde{K}} \widetilde{\xi}(t) &= \underbrace{\begin{bmatrix} 0 \\ \widehat{\mathcal{B}}_6 \end{bmatrix}}_{=:\widetilde{B}} u(t), \\ \widetilde{y}(t) &= \widetilde{B}^\top \dot{\widetilde{\xi}}(t). \end{aligned}$$

*Remark* 8.1. In the case that the submatrix $\widetilde{\mathcal{A}}_{\mathrm{z}}$ of $\widetilde{\mathcal{A}}$ as in (5.8) does not have semisimple eigenvalues, one can perturb the blocks of $\widetilde{\mathcal{A}}_{\mathrm{z}}$ corresponding to positive real characteristic values that are smaller than one. In doing so, if additionally $\widetilde{\mathcal{C}}\widetilde{\mathcal{A}}\widetilde{\mathcal{B}} > 0$, the newly formed system $[\widetilde{\mathcal{A}} + \Delta, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}]$, for some sufficiently small $\|\Delta\|_2$, will then still be passive (see [5]), and the quality of the reduced system is not affected.

We illustrate the performance of the above procedure with an example of three coupled mass-spring-damper chains; see [39, Ex. 2].

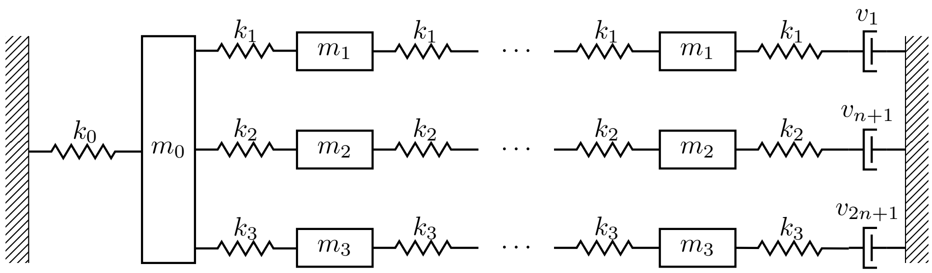



FIG. 8.1. *Triple chain oscillator with* $(3n+1)$ *masses and three dampers.*

*Example* 8.2. The triple chain consists of three rows that are coupled via a mass $m_0$ which is connected to the fixed base with a spring with stiffness $k_0$. Each row contains $n$ masses, $n+1$ springs, and one damper. The latter is attached to a wall; see Figure 8.1. One can write the free system as

$$M\ddot{\xi}(t) + D\dot{\xi}(t) + K\xi(t) = 0,$$

where $M, D$, and $K$ are defined as $M = \operatorname{diag}(m_1, \ldots, m_1, m_2, \ldots, m_2, m_3, \ldots, m_3, m_0)$ and $D = \alpha M + \beta K + v(e_1 e_1^\top + e_{n+1}e_{n+1}^\top + e_{2n+1}e_{2n+1}^\top)$, where $e_i$ denotes the $i$th unit vector in $\mathbb{R}^n$ and with the dampers' viscosity $v$. Moreover,

$$K = \begin{bmatrix} K_{11} & & & -\kappa_1 \\ & K_{22} & & -\kappa_2 \\ & & K_{33} & -\kappa_3 \\ -\kappa_1^\top & -\kappa_2^\top & -\kappa_3^\top & k_1+k_2+k_3+k_0 \end{bmatrix}, \quad K_{ii} = k_i \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix},$$

where $\kappa_i = \begin{bmatrix} 0 & \ldots & 0 & k_i \end{bmatrix}^\top \in \mathbb{R}^{1\times n}$ and $K_{ii} \in \mathbb{R}^{n\times n}$ for $i = 1, 2, 3$. We choose the input $b = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}^\top$ and equally measure the velocities $c_{\mathrm{v}} = \begin{bmatrix} 1 & \ldots & 1 \end{bmatrix}$. The second order control system then reads

$$M\ddot{\xi}(t) + D\dot{\xi}(t) + K\xi(t) = bu(t), \quad y(t) = c_{\mathrm{v}}\dot{\xi}(t).$$

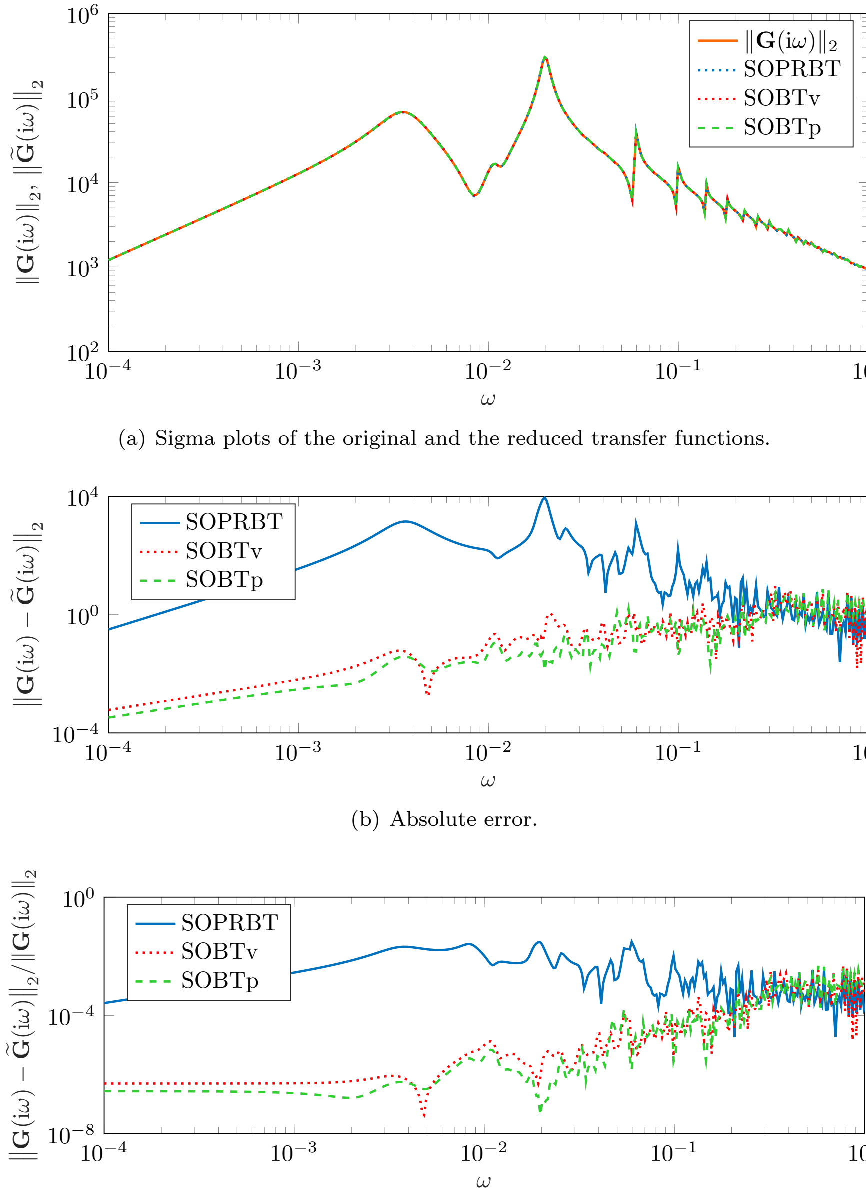
We consider the triple chain with $n = 500$; thus the number of positions is $3n+1 = 1501$, $k_0 = 50$, $k_1 = 10$, $k_2 = 20$, $k_3 = 1$, $m_0 = 1$, $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $\alpha = \beta = 0.002$, and $v = 5$. Following the previously presented numerical procedure we first compute a first order reduced model of order $2r = 200$. Applying (2) in this procedure, we obtain a second order model of order $r = 100$. The latter has the form

$$\widetilde{M}\ddot{\widetilde{\xi}}(t) + \widetilde{D}\dot{\widetilde{\xi}}(t) + \widetilde{K}\widetilde{\xi}(t) = \widetilde{B}u(t), \quad \widetilde{y}(t) = \widetilde{B}^\top \dot{\widetilde{\xi}}(t),$$

with symmetric $\widetilde{M}$, $\widetilde{D}$, $\widetilde{K} \in \mathbb{R}^{r\times r}$, where $\widetilde{M} = I_r$, $\widetilde{K} > 0$, and $\widetilde{D} = \widetilde{D}^\top$ and $\widetilde{B} \in \mathbb{R}^{r\times m}$.

Next we present numerical results computed with MATLAB R2020a on a machine with an AMD Ryzen 7 3700X 3.60 GHz 8-core processor and 16 GB of RAM. The involved Riccati and Lyapunov equations have been solved using an unpublished preliminary version of the M-M.E.S.S. 2.0.1 toolbox[1] [35] with a maximum of 300 ADI iterations.

[1] Available from https://gitlab.mpi-magdeburg.mpg.de/mess/mmess-releases, GIT repository tag `bc7be0cd5154201b3f988ef3a52f119406e173e7`.




(a) Sigma plots of the original and the reduced transfer functions.

(b) Absolute error.

(c) Relative error.

FIG. 8.2. *Frequency plots of original and reduced transfer functions and errors.*

The sigma plots of the original and reduced transfer functions $\mathbf{G}(s)$ and $\widetilde{\mathbf{G}}(s)$ for the reduced models received from our method (SOPRBT), velocity balanced truncation (SOBTv) from [33], and position balanced truncation (SOBTp) from [33, 42] are depicted in Figure 8.2(a), whereas Figures 8.2(b) and 8.2(c) show the absolute and the relative error of the transfer functions evaluated on the imaginary axis for these methods. We choose SOBTv and SOBTp as a comparison, since for symmetric models with velocity measurements of the form (1.1) these methods are also structure preserving. With a maximum relative error of approximately $3.1 \cdot 10^{-2}$ for our method we obtain a good match between the original and the reduced second order system, which is comparable to the maximum errors of velocity balanced truncation, approximately $2.7 \cdot 10^{-2}$, and position balanced truncation, approximately $2.0 \cdot 10^{-2}$. Admittedly, the maximum absolute error of our method, reaching $9.1 \cdot 10^{+3}$, is higher than those of velocity and position balanced truncation for this numerical example, which are approximately $1.9 \cdot 10^{+1}$ and $1.5 \cdot 10^{+1}$, respectively. In this example the structural error from (1)(d) of our numerical procedure is $\varepsilon_{\text{struct}} = 3.7 \cdot 10^{-2}$, while the gap error bound and the relative gap error bound, estimated as in (1)(c) are 4.8 and $3.9 \cdot 10^{-2}$, respectively. Concerning computation times, executing SOPRBT took $5.10s$, while SOBTv took $1.71s$, and SOBTp took $1.58s$.

**9. Conclusion and outlook.** We have presented a numerical procedure for model reduction of passive second order systems that preserves asymptotic stability and passivity and restores the second order structure. The underlying reduction method is positive real balanced truncation, which provides the gap metric error bound from [19] and is shown to yield a first order system with a special structure. It is further shown that, under some further conditions, this reduced order system can be represented as a second order system.

An extension of these methods to systems with position measurements and non-co-located inputs and outputs is planned for future work. Even though these systems are in general not passive, one can employ similar methods using artificial inputs and outputs that lead to passive extensions.

**Code and data availability.** The MATLAB code and data for reproducing the numerical results presented in this paper are available through doi:10.5281/zenodo.4519721.

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